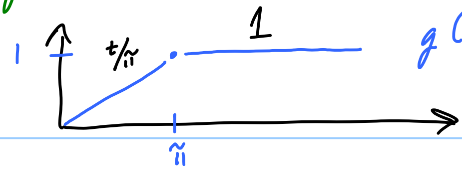
Lesson 27 Using Laplace transform to solve IVPs.

$y'' + y =$ 

$g(t) = \frac{t}{\pi}\left(u_0(t) - u_\pi(t)\right) + 1 \cdot u_\pi(t)$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$(s^2+1)Y = G(s) = \frac{1}{\pi}\cdot\frac{1}{s^2} - \frac{1}{\pi} e^{-\pi s}\cdot\frac{1}{s^2}$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\text{Transfer fcn}} \cdot G(s) = \underbrace{\frac{1}{\pi}\,\frac{1}{s^2(s^2+1)}}_{H(s)} - e^{-\pi s}\left[\underbrace{\frac{1}{\pi}\,\frac{1}{s^2(s^2+1)}}_{H(s)}\right]$$

$$Y = H(s) - e^{-\pi s} H(s)$$

$$\mathcal{L}\left[u_c(t) f(t-c)\right] = e^{-cs} F(s)$$

So $\quad y = h(t) - u_\pi(t)\, h(t-\pi)$

$$\pi H(s) = \frac{1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}$$

Mult. by denom $s^2(s^2+1)$: $\quad 1 = A(s^2+1) + Bs(s^2+1) + (Cs+D)s^2$

$$0\cdot s^3 + 0\cdot s^2 + 0s + 1 = \underbrace{(B+C)}_{0}s^3 + \underbrace{(A+D)}_{0}s^2 + \underbrace{(B)}_{0}s + \underbrace{(A)}_{1}$$

$A = 1$
$B = 0$
$D = -1$
$C = 0$

So $\quad H(s) = \frac{1}{\pi}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$ and

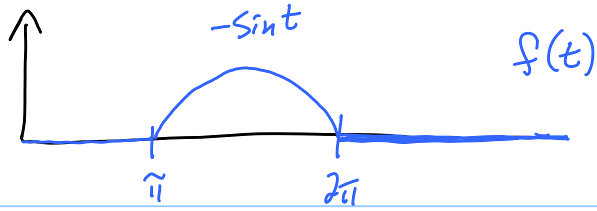
$h(t) = \frac{1}{\pi}(t - \sin t)$ and

$$y(t) = \frac{1}{\pi}(t - \sin t) - \frac{1}{\pi} u_\pi(t)\left[(t-\pi) - \sin(t-\pi)\right]$$

$$= \begin{cases} \frac{1}{\pi}(t - \sin t) & 0 \le t \le \pi \\ 1 - \frac{2}{\pi}\sin t & t \ge \pi \end{cases}$$

$\leftarrow \sin(t-\pi) = -\sin t$

Tricks:



$$f(t) = (-\sin t)\left[u_{\pi}(t) - u_{2\pi}(t) \right]$$

On at π Off at 2π

$\mathcal{L}\left[u_c(t)f(t-c)\right] = e^{-cs}F(s)$

$$= -u_{\pi}(t)\sin t + \underbrace{u_{2\pi}\sin t}_{u_{2\pi}(t)\sin(t-2\pi)}$$

$u_{2\pi}(t)\sin(t-2\pi) \leftarrow$ because $\sin$ is 2π periodic

$$\mathcal{L} = e^{-2\pi s}\cdot\frac{1}{s^2+1}$$

$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$

Hmmm: $\sin(t) = \sin\left((t-\pi)+\pi\right)$

$$= \cos(t-\pi)\underbrace{\sin\pi}_{0} + \sin(t-\pi)\underbrace{\cos\pi}_{-1} = -\sin(t-\pi)$$

So $f(t) = u_{\pi}(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi)$ and

$$F(s) = e^{-\pi s}\cdot\frac{1}{s^2+1} + e^{-2\pi s}\cdot\frac{1}{s^2+1}$$

EX: $g(t) = u_3(t)e^{2t} = u_3(t)\,e^{2((t-3)+3)}$

$$= e^{6}u_3(t)\underbrace{e^{2(t-3)}}$$

$f(t-3) = e^{2(t-3)}$

$f(\ ) = e^{2(\ )}$

$f(t) = e^{2t}$ so $F(s) = \frac{1}{s-2}$

Finally $G(s) = e^{6}\cdot e^{-3s}F(s)$

$$= e^{-3s}\cdot\frac{e^{6}}{s-2}$$

Partial Fractions:

$$\frac{s^{13}+s^2+7}{s^3(s-1)^4(s-3)\underbrace{(s^2+2s+2)}_{\text{complex roots}}(s^2+1)^2}$$

$\leftarrow$ deg < 14!

$\leftarrow$ deg 14

(if not, Long Division)

$$= \underbrace{\left(\frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s}\right)}_{\text{for } s^3 = (s-0)^3} + \underbrace{\left(\frac{D}{(s-1)^4} + \frac{E}{(s-1)^3} + \frac{F}{(s-1)^2} + \frac{G}{s-1}\right)}_{\text{for } (s-1)^3} + \underbrace{\frac{H}{s-3}}_{\text{for } s-3} +$$

$$+ \underbrace{\frac{Is+J}{s^2+2s+2}}_{\text{for } s^2+2s+2} + \underbrace{\left(\frac{Ks+L}{(s^2+1)^2} + \frac{Ms+N}{(s^2+1)}\right)}_{\text{for } (s^2+1)^2}$$

Tricks: $\mathcal{L}[t \sin t] = \cancel{\mathcal{L}[t]\,\mathcal{L}[\sin t]}$ No, no, no!

$$= \int_0^\infty e^{-st}\, t \sin t \, dt$$ Ouch!

$f(t) = t\sin t$

$f'(t) = t\cos t + \sin t$

$f''(t) = -t\sin t + 2\cos t$

$F(s) = \mathcal{L}[f(t)]$

Aha!

$$\mathcal{L}[f''] = s^2 F(s) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{0}$$

but

$$\mathcal{L}[f''] = \mathcal{L}[-t\sin t] + \mathcal{L}[2\cos t]$$

$$= -\mathcal{L}[f] + \frac{2s}{s^2+1}$$

Finally

$$s^2 F(s) = -F(s) + \frac{2s}{s^2+1}$$ ← solve for $F(s)$

$$F(s) = \frac{2s}{(s^2+1)^2}$$

So $\mathcal{L}[t\sin t] = \dfrac{2s}{(s^2+1)^2}$

Easier example: $f(t) = te^{at}$ $\qquad \mathcal{L}[f] = F$

$f'(t) = ate^{at} + e^{at} = af(t) + e^{at}$

$$\mathcal{L}[f'] = a\mathcal{L}[f] + \mathcal{L}[e^{at}] = aF(s) + \frac{1}{s-a}$$

and $\mathcal{L}[f'] = sF(s) - \underbrace{f(0)}_{0} = sF(s)$

Two ways to write $\mathcal{L}[f'] = \begin{cases} aF(s) + \frac{1}{s-a} \\ sF(s) \end{cases}$)) equate!

Equate, solve for $F(s)$: $F(s) = \frac{1}{(s-a)^2}$

Dumb example because of s-shift rule $\mathcal{L}[f] = F(s)$

$$\mathcal{L}\left[e^{at} \underbrace{f(t)}_{t}\right] = F(s-a)$$

$$\mathcal{L}\left[te^{at}\right] = \frac{1}{(s-a)^2} \checkmark$$