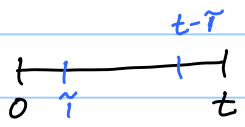


Lesson 29: Convolution 6.6

$$\mathcal{L}[t \sin t] = \cancel{\frac{1}{s^2}} \cdot \cancel{\frac{1}{s^2+1}} \quad \text{no!}$$

True: $\mathcal{L}[f * g] = F(s) \cdot G(s)$

where $(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau$



The diagram shows a horizontal line segment from 0 to t. A point \tau is marked on the segment, and the remaining length from \tau to t is labeled t - \tau.

[Discrete version $\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n$

where $c_n = \sum_{k=0}^n a_k b_{n-k}$

EX: Find $\mathcal{L}^{-1} \left[\frac{1}{(s^2+1)^2} \right]$

$$\mathcal{L} \left[\underset{\substack{\uparrow \\ \sin t}}{f} * \underset{\substack{\uparrow \\ \sin t}}{g} \right] = \underbrace{F(s)}_{\substack{\uparrow \\ \frac{1}{s^2+1}}} \cdot \underbrace{G(s)}_{\substack{\uparrow \\ \frac{1}{s^2+1}}}$$

$$\text{Ans: } (f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau = \int_0^t \underbrace{\sin(\tau)}_{\alpha} \underbrace{\sin(t - \tau)}_{\beta} d\tau$$

$$\sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta)$$

$$\begin{aligned} \sin \tau \sin(t - \tau) &= \frac{1}{2} \cos[\tau - (t - \tau)] - \frac{1}{2} \cos(\tau + (t - \tau)) \\ &= \frac{1}{2} \cos(2\tau - t) - \frac{1}{2} \cos t \end{aligned}$$

$$\begin{aligned} \text{Ans: } & \frac{1}{2} \int_0^t \underbrace{\cos(2\tau - t)}_u - \cos t \, d\tau \\ &= \frac{1}{2} \int_{u=-t}^t \cos u \left(\frac{1}{2} du \right) - \frac{1}{2} \cos t \underbrace{\int_0^t 1 \, d\tau}_t \end{aligned}$$

$u = 2\tau - t \quad \left| \begin{array}{l} \text{When } \tau = 0: u = -t \\ \tau = t: u = t \end{array} \right.$
 $du = 2d\tau$

$$= \frac{1}{4} \sin u \Big|_{u=-t}^{u=t} - \frac{1}{2} t \cos t$$

$$= \frac{1}{4} \left(\sin t - \underbrace{\sin(-t)}_{-\sin t} \right) - \frac{1}{2} t \cos t$$

$$= \frac{1}{2} \sin t - \frac{1}{2} t \cos t$$

Remark: $\mathcal{L}[t f(t)] = -F'(s) \leftarrow \begin{matrix} \text{cousin} \\ \text{formula} \\ \text{to} \end{matrix} \mathcal{L}[y'] = sY - y(0)$

So $\mathcal{L}[t \cos t] = -\frac{d}{ds} \left[\frac{s}{s^2+1} \right]$

Exciting thing: $y'' + y = r(t) \quad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$(s^2+1)Y = R(s)$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}[\sin(t)]} \cdot \underbrace{R(s)}_{\mathcal{L}[r(t)]}$$

$$\text{So } y = (r * \sin)(t) = \int_0^t r(\tau) \sin(t-\tau) d\tau$$

Wow!

Fantastic formula if r is a measured function.

Remark: $f * g = g * f$

Why: $F(s) \cdot G(s) = G(s) \cdot F(s)$. Both are $\mathcal{L}^{-1}$ of this!
($\mathcal{L}^{-1}$ is unique!)

Why: $\mathcal{L}[(f * g)(t)] = F(s) \cdot G(s) = \int_0^\infty f(u) e^{-s(u)} du \int_0^\infty g(v) e^{-s(v)} dv$

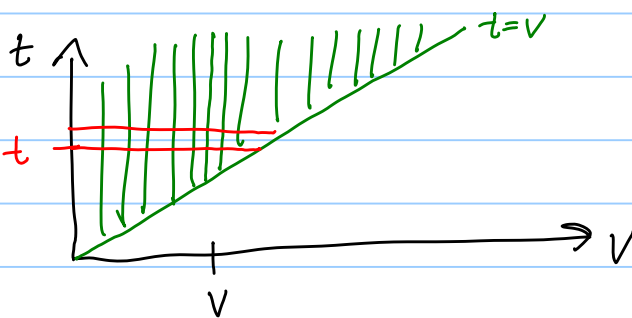
$$= \int_{v=0}^{\infty} \left[\int_{u=0}^v e^{-s(u+v)} f(u) g(v) du \right] dv$$

$$= \int_{v=0}^{\infty} g(v) \left[\int_{u=0}^v e^{-s(\underbrace{u+v}_t)} f(u) du \right] dv$$

$$\begin{aligned} t &= u+v \\ dt &= du \end{aligned}$$

$$\begin{aligned} \text{When } u=0 : t &= v \\ u \rightarrow \infty : t &\rightarrow \infty \end{aligned}$$

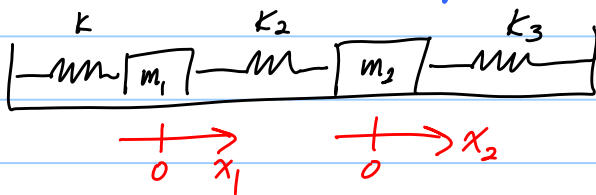
$$= \int_{v=0}^{\infty} g(v) \left[\int_{t=v}^{\infty} e^{-st} f(t-v) dt \right] dv$$



$$= \int_{t=0}^{\infty} e^{-st} \underbrace{\left(\int_{v=0}^t g(v) f(t-v) dv \right)}_{(g * f)(t)} dt$$

Fubini's Thm

Friday: Start systems. Chap. 7



$$m's, k's = 1.$$

$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 \\ \ddot{x}_2 = x_1 - 2x_2 \end{cases} \quad \text{Hit it with } \mathcal{L}!$$

$$\begin{cases} s^2 \bar{x}_1 - s x_1'(0) - x_1(0) = -2 \bar{x}_1 + \bar{x}_2 \\ s^2 \bar{x}_2 - s x_2'(0) - x_2(0) = \bar{x}_1 - 2 \bar{x}_2 \end{cases}$$

$$\begin{pmatrix} -s \dot{x}_1(0) - x_1(0) \\ -s \dot{x}_2(0) - x_2(0) \end{pmatrix} = \begin{bmatrix} -2-s^2 & 1 \\ 1 & -2-s^2 \end{bmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

use Cramer's, Easy