Lesson 29: Convolution 6.6

$$\mathcal{L}[t\sin t] = \cancel{\frac{1}{s^2}\cdot\frac{1}{s^2+1}}$$ no!

True: $\mathcal{L}[f*g] = F(s)\cdot G(s)$

where $(f*g)(t) = \int_0^t f(\tilde{\tau})\,g(t-\tilde{\tau})\,d\tilde{\tau}$

0 $\quad\tau\quad$ $t-\tau$ $\quad$ t

[Discrete version $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

EX: Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$

$$\mathcal{L}[\underset{\sin t}{f} * \underset{\sin t}{g}] = \underbrace{F(s)}_{\frac{1}{s^2+1}}\cdot \underset{\frac{1}{s^2+1}}{G(s)}$$

Ans: $(f*g)(t) = \int_0^t f(\tilde{\tau})\,g(t-\tilde{\tau})\,d\tilde{\tau} = \int_0^t \sin(\underbrace{\tilde{\tau}}_{\alpha})\sin(\underbrace{t-\tilde{\tau}}_{\beta})\,d\tilde{\tau}$

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

$$\sin\tilde{\tau}\sin(t-\tilde{\tau}) = \frac{1}{2}\cos\left[\tilde{\tau}-(t-\tilde{\tau})\right] - \frac{1}{2}\cos(\tilde{\tau}+(t-\tilde{\tau}))$$

$$= \frac{1}{2}\cos(2\tilde{\tau}-t) - \frac{1}{2}\cos t$$

Ans: $\frac{1}{2}\int_0^t \cos(\underbrace{2\tilde{\tau}-t}_{u}) - \cos t\ \ d\tilde{\tau}$

$u = 2\tilde{\tau} - t$ | When $\tilde{\tau}=0$: $u=-t$

$du = 2\,d\tilde{\tau}$ | $\tilde{\tau}=t$: $u=t$

$$= \frac{1}{2}\int_{u=-t}^{t} \cos u\left(\frac{1}{2}du\right) - \frac{1}{2}\cos t\underbrace{\int_0^t 1\,d\tilde{\tau}}_{t}$$

$$= \frac{1}{4} \sin u \Big]_{u=-t}^{u=t} - \frac{1}{2} t \cos t$$

$$= \frac{1}{4}\Big(\sin t - \underbrace{\sin(-t)}_{-\sin t}\Big) - \frac{1}{2} t \cos t$$

$$= \frac{1}{2} \sin t - \frac{1}{2} t \cos t$$

Remark: $\mathcal{L}[t f(t)] = -F'(s)$ ← cousin formula to $\mathcal{L}[y'] = sY - y(0)$

So $\mathcal{L}[t \cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$

Exciting thing: $y'' + y = r(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$(s^2+1) Y = R(s)$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}[\sin(t)]} \cdot \underbrace{R(s)}_{\mathcal{L}[r(t)]}$$

So $y = (r * \sin)(t) = \underline{\underline{\int_0^t r(\tau) \sin(t-\tau)\, d\tau}}$

Wow!

Fantastic formula if r is a measured function.

Remark: $f * g = g * f$

Why: $F(s) \cdot G(s) = G(s) \cdot F(s)$. Both are $\mathcal{L}^{-1}$ of this!

($\mathcal{L}^{-1}$ is unique!)

Why: $\mathcal{L}[(f*g)(t)] = F(s) \cdot G(s) = \int_0^\infty f(u) e^{-s(u)}\, du \int_0^\infty g(v) e^{-s(v)}\, dv$

$$= \int_{v=0}^{\infty} \left[\int_{u=0}^{\infty} e^{-s(u+v)} f(u)\, g(v) \; du \right] dv$$

$$= \int_{v=0}^{\infty} g(v) \left[\int_{u=0}^{\infty} e^{-s(\underbrace{u+v}_{t})} f(u)\, du \right] dv$$

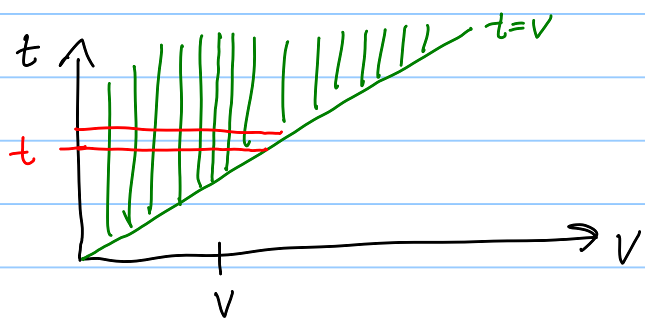
$t = u+v$

$dt = du$

When $u=0$: $t = v$

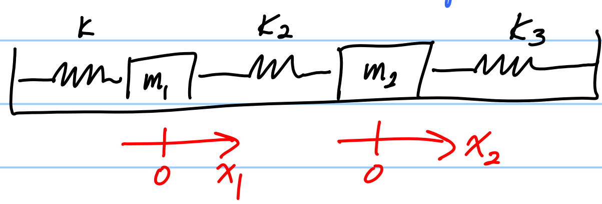
$u \to \infty$: $t \to \infty$

$$= \int_{v=0}^{\infty} g(v) \left[\int_{t=v}^{\infty} e^{-st} f(t-v)\, dt \right] dv$$



$$\underset{\text{Fubinni's Thm}}{=} \int_{t=0}^{\infty} e^{-st} \Bigg(\underbrace{\int_{v=0}^{t} g(v) f(t-v)\, dv}_{(g*f)(t)} \Bigg) dt$$

Friday: Start systems. Chap. 7



m's, k's $= 1$.

$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 \\ \ddot{x}_2 = x_1 - 2x_2 \end{cases}$$

Hit it with $\mathcal{L}$!

$$\begin{cases} s^2 X_1 - s x_1'(0) - x_1(0) = -2X_1 + X_2 \\ s^2 X_2 - s x_2'(0) - x_2(0) = X_1 - 2X_2 \end{cases}$$

$$\left\{ \begin{pmatrix} -s\dot{x}_1(0) - x_1(0) \\ -s\dot{x}_2(0) - x_2(0) \end{pmatrix} = \begin{bmatrix} -2-s^2 & 1 \\ 1 & -2-s^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right.$$

use Cramer's, Easy