Lesson 32: Linear First order homogeneous 2x2 systems with constant coefficients

Last time: $\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases}$ $\qquad \vec{x}\,' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{x}$

Char. Eqn

$\det(\mathbb{A} - r\,\mathbb{I}) = 0$

$r = -2 \pm 2i$

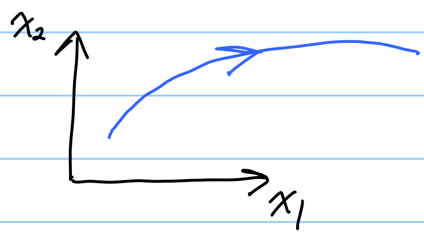
$$\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$$

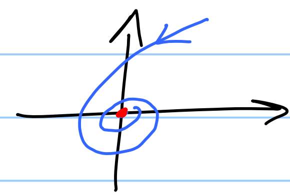
Check Wronskian: $W(0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

Fact: The Wronskian is either $\equiv 0$ or never zero.

Abel's Formula: $W = C e^{\int p_{11} + p_{22} \, dt}$

Phase plane:



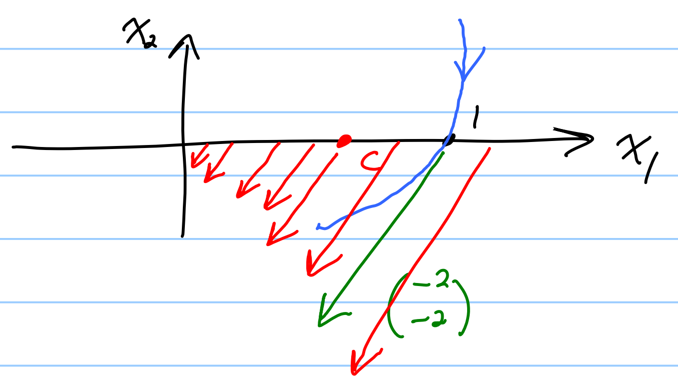


Our sol^n is a spiral that goes in.

$(0,0)$ is a Critical Point. $\begin{cases} x_1 \equiv 0 \\ x_2 \equiv 0 \end{cases}$ is a sol^n.

Word: $(0,0)$ is an asymptotically stable critical point.

Clockwise or counterclockwise? Look at field along positive x_1 axis.



$\vec{x}\,' = \mathbb{A}\vec{x}$

$\vec{x}\,'$ at $\begin{pmatrix} c \\ 0 \end{pmatrix}$ is $\mathbb{A}\begin{pmatrix} c \\ 0 \end{pmatrix} =$

$\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} c$

It is clockwise. In.

Why a complex sol^n gives two real sol^ns: $\vec{x} = \vec{u} + i\vec{v}$ is a comp sol^n to $\vec{x}' = A\vec{x}$

$$(\vec{u} + i\vec{v})' = A(\vec{u} + i\vec{v})$$

$$\vec{u}' + i\vec{v}' = A\vec{u} + iA\vec{v}$$

$$[\underbrace{\vec{u}' - A\vec{u}}_{\text{zero vector}}] = i\,[\underbrace{-\vec{v}' + A\vec{v}}_{=\vec{0}}] \qquad \begin{pmatrix} a \\ b \end{pmatrix} = i\begin{pmatrix} c \\ d \end{pmatrix}$$

EX: $\begin{pmatrix} 3+5i \\ 4+6i \end{pmatrix} e^{(2+3i)t} = \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix} + i\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right]\left(e^{2t}\cos 3t + i e^{2t}\sin 3t\right)$

$$\underbrace{e^{2t}\left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}\cos 3t \underset{i^2=-1}{-} \begin{pmatrix} 5 \\ 6 \end{pmatrix}\sin 3t\right]}_{\vec{x}_1 \text{ a real } sol^n} + i\underbrace{\left[\begin{pmatrix} 5 \\ 6 \end{pmatrix}\cos 3t + \begin{pmatrix} 3 \\ 4 \end{pmatrix}\sin 3t\right]e^{2t}}_{\vec{x}_2\text{, real } sol^n}$$

Fact: Complex roots to $\det(A - rI) = 0$.

$$r = a \pm bi \qquad e^{(a+bi)} = e^{at}(\cos bt + i\sin bt)$$

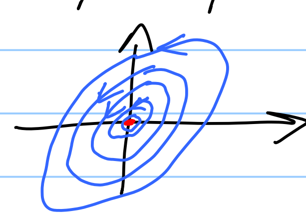
Aha! $a < 0$ Spiral in.

$a > 0$ Spiral out

What happens when $a = 0$? Trajectories in phase plane are ellipses.

Critical point is called a Center.

It is just plain stable.



EX: $\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$

$\det(A - rI) = 0$

$\det \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} = 0$

$(5-r)(1-r) - (-3) = 0$

$r^2 - 6r + 8 = 0$

$(r-2)(r-4) = 0$

$r = 2, 4$

For $r=2$: $(A - 2I)\vec{a} = \vec{0}$

$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \vec{a} = \vec{0}$

$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Get $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$

For $r=4$: $\left[\begin{array}{cc|c} 5-4 & -1 & 0 \\ 3 & 1-4 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

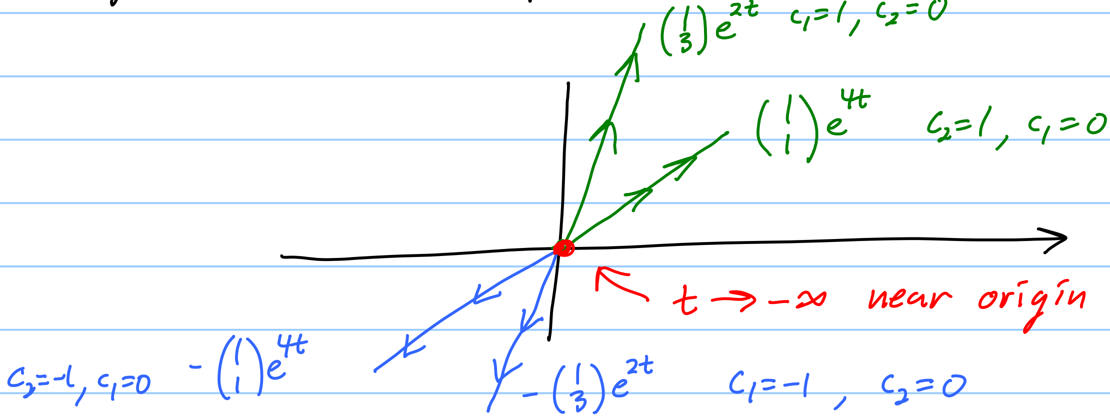
$\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

Gen'l Sol'n: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

All sol'n's $\to$ "∞" as $t \to \infty$.

Origin is called an <u>improper node</u>.

Origin is <u>unstable</u>.

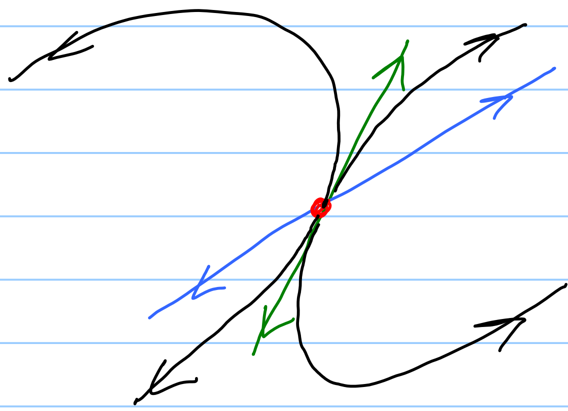


As $t \to -\infty$ e^{4t} is a hell of a lot smaller than e^{2t}.

Solutions leave origin tangent to $\begin{pmatrix}1\\3\end{pmatrix}$.

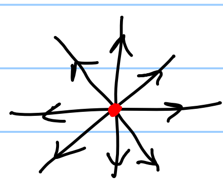
Why: $\vec{x} = c_1 \begin{pmatrix}1\\3\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{4t}$

$$= e^{2t}\left[\underset{\substack{\uparrow\\ \text{sliding in along } \pm\begin{pmatrix}1\\3\end{pmatrix}.}}{c_1 \begin{pmatrix}1\\3\end{pmatrix}} + \underbrace{c_2 \begin{pmatrix}1\\1\end{pmatrix} e^{2t}}_{\text{dies away}}\right] \leftarrow \text{let } t \to -\infty$$



Improper node.
Unstable.

Proper node:



$$\begin{cases} \dfrac{dx_1}{dt} = k_1 x_1 \\[2ex] \dfrac{dx_2}{dt} = k_2 x_2 \end{cases}$$

"Uncoupled"

Improper nodes: Roots r_1, r_2 are real and same sign.

Case $0 < r_1 < r_2$: Unstable

Case $r_2 < r_1 < 0$: Asymptotically stable.

Node rule: Trajectories near the origin hug $(\pm)$ the eigenvector corresponding to the eigenvalue closest to zero.

EX: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}$

$r = -1 \leftarrow$ closest to zero

$r = -2$