Lesson 34 Repeated eigenvalues

3x3 system: $\quad \vec{x}\,' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix} \vec{x}$

$\det(\mathbb{A} - r\,\mathbb{I}) = 0 \qquad$ Got $r = -1, \pm 2i$.

For $r=-1$: $\quad$ Got $\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$

For $r = 2i$: $\quad (\mathbb{A} - (2i)\mathbb{I})\,\vec{a} = 0$

$$\begin{bmatrix} -2i & 4 & 0 \\ -1 & -2i & 0 \\ 1 & 4 & -1-2i \end{bmatrix} \vec{a} = \vec{0}$$

MAPLE $\rightsquigarrow$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ a_1 \text{ bound} & a_2 \text{ bound} & a_3 \text{ free!} \end{array} \end{array}$$

Gaussian elimination
(rref
"row reduced echelon form")

$$\begin{cases} a_1 - a_3 = 0 \\ a_2 - \frac{i}{2} a_3 = 0 \end{cases}$$

Let $a_3 = c$, an arbitrary constant

$$\begin{cases} a_1 = a_3 = c \\ a_2 = \frac{i}{2} a_3 = \frac{i}{2} c \\ a_3 = c \end{cases}$$

So $\quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c \\ \frac{i}{2} c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ i/2 \\ 1 \end{pmatrix} c$

Nice non-zero sol^n: Take $c = 2$. Get $\vec{a} = \begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix}$.

Get complex sol^n: $\quad \vec{x} = \begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix} e^{2it} = \left[\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] (\cos 2t + i \sin 2t)$

$$= \underbrace{\left[\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cos 2t \underset{\substack{\uparrow \\ i^2}}{-} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t \right]}_{\vec{x}_2} + i \underbrace{\left[\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t \right]}_{\vec{x}_3}$$

Gen^l Sol^n:

$$\vec{x} = c_1\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}2\cos 2t\\-\sin 2t\\2\cos 2t\end{pmatrix} + c_3\begin{pmatrix}2\sin 2t\\\cos 2t\\2\sin 2t\end{pmatrix}$$

Wronskian: $W(0) = \det\begin{bmatrix}0 & 2 & 0\\0 & 0 & 1\\1 & 2 & 0\end{bmatrix} = 2 \neq 0$

So W is never zero. Gen^l Sol^n

$$\begin{cases} x_1 = 2c_2\cos 2t + 2c_3\sin 2t\\ x_2 = -c_2\sin 2t + c_3\cos 2t\\ x_3 = c_1e^{-t} + 2c_2\cos 2t + 2c_3\sin 2t\end{cases}$$

EX: Repeated roots $\quad \vec{x}' = \begin{bmatrix}-3 & 1\\-1 & -1\end{bmatrix}\vec{x}$

$$\det(A - rI) = \det\begin{bmatrix}-3-r & 1\\-1 & -1-r\end{bmatrix} = (-3-r)(-1-r)+1$$

$$= r^2 + 4r + 4$$

$$= (r+2)^2 = 0$$

Only get one r. $\boxed{r=-2}$.

For $r=-2$: $(A-(-2)I)\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c}-3-(-2) & 1 & 0\\-1 & -1-(-2) & 0\end{array}\right]$$

$$\left[\begin{array}{cc|c}-1 & 1 & 0\\-1 & 1 & 0\end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a_1 bound $\quad$ a_2 free $\quad$ Let $a_2 = c$, arbitrary.

$$a_1 - a_2 = 0$$

So $\quad a_1 = a_2 = c$

$$\begin{cases} a_1 = c \\ a_2 = c \end{cases} \qquad \vec{a} = \begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} c$$

Nice one: Take $c = 1$. Get sol^n $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$.

Need $\vec{x}_2$! Try t times $\vec{x}_1$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-2t}$. FAILS!

Better idea: Try $\vec{x}_2 = \vec{a}te^{rt} + \vec{b}e^{rt}$ and plug it in and make it work.

$$\vec{x}_2{}' = \vec{a}e^{rt} + r\vec{a}te^{rt} + r\vec{b}e^{rt} \overset{\text{want}}{=} A\left[\vec{a}te^{rt} + \vec{b}e^{rt}\right]$$

$$\Big[\underbrace{r\vec{a} - A\vec{a}}_{\text{need } = 0}\Big]te^{rt} = \Big[\underbrace{A\vec{b} - r\vec{b} - \vec{a}}_{\text{need } = 0}\Big]e^{rt}$$

Need $\quad A\vec{a} = r\vec{a}$ ← Aha! $\vec{a}$ has to be the e-vect for $r = -2$.

$$\boxed{(A - rI)\vec{b} = \vec{a}}$$

← Linear algebra: This problem has sol^{n}s $\vec{b}$ even though det $= 0$!

We have $r = -2$, $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$(A - (-2)I)\vec{b} = \vec{a}$$

$$\left[\begin{array}{cc|c} -3-(-2) & 1 & 1 \\ -1 & -1-(-2) & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{bound} & \text{free} \end{array}$$

Let $b_2 = c$.

$$b_1 - b_2 = -1$$

$$\begin{cases} b_1 = b_2 - 1 = c - 1 \\ b_2 = c \end{cases}$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} c-1 \\ c \end{pmatrix}.$$

Take $c = 0$.

Get $\vec{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

$$\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-2t}$$

Gen^l Sol^n $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} te^{-2t} - e^{-2t} \\ te^{-2t} \end{pmatrix}$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} t-1 \\ t \end{pmatrix} e^{-2t}$$