Lesson 35 Non-homogeneous systems, Variation of Parameters

EX: $\begin{cases} \dfrac{dx_1}{dt} = x_1 \\ \dfrac{dx_2}{dt} = x_2 \end{cases}$ $\quad \vec{x}\,' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$

(1): $x_1 = c_1 e^t$

(2): $x_2 = c_2 e^t$

$\vec{x} = \binom{c_1}{c_2} e^t = c_1 \binom{1}{0} e^t + c_2 \binom{0}{1} e^t$

$\det(A - r\mathbb{I}) = (1-r)^2 = 0$

$r = 1, 1 \leftarrow$ double root!

$(A - (1)\mathbb{I})\vec{a} = \vec{0}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{a} = \vec{0}$!

All solutions called the eigenspace for eval $r=1$.

This e-space is 2 dim^e!

$\vec{a} = \binom{c_1}{c_2} = c_1 \binom{1}{0} + c_2 \binom{0}{1}$

Get 2 sol^{n}s: $\binom{1}{0} e^t$ and $\binom{0}{1} e^t$

Multiplicity of root r of $\det(A - r\mathbb{I}) = 0$ is called the algebraic multiplicity of the e-value. The dimension of the e-space for e-val r is called the geometric multiplicity.

Fact: (geom. mult) $\leq$ (alg. mult).

What does $\det(A) = 0$ mean?

2x2 case: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 0$ $\qquad ad = bc$

$\dfrac{a}{c} = \dfrac{b}{d} = K$

First row $= K$ times the second row ... if $c, d \neq 0$.

Possibilities: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} \text{~~~} & \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 \\ \text{~~~} & \end{bmatrix}$,

or one row is a multiple of the other.

Hmmm, but $[0,0]$ is a multiple of [illegible]. $c = 0$.

So $\det(2\times 2) = 0$ means one row = mult of the other.

Word: Rows are linearly dependent.

Linearly independent means rows not parallel. They form a coordinate system for $\mathbb{R}^2$.

In $\mathbb{R}^n$: $\mathbb{A}$ is $n\times n$. $\det(\mathbb{A}) = 0$ means rows are lin. dependent.

Def^n: Vectors lin. independent means:

$$\text{If} \quad c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0}$$

then all the c's must $= 0$.

Consequently: Vectors are lin dependent if one is a linear combo of the others.

The Wronskian connection: $\vec{x}\,' = \mathbb{A}\vec{x}$, $\mathbb{A}$ $n\times n$.

$\vec{x} = c_1\vec{x}_1 + \cdots + c_2\vec{x}_n$ is the gen^l sol^n exactly

when sol^{n}s $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent.

Non-homogeneous system:

$$\vec{x}\,' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{x} + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}}_{\vec{g}}$$

$$\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g}$$

Step 1: Solve homog syst: $\vec{x}\,' = \mathbb{A}\vec{x}$

Get $\vec{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$ by usual method.

<u>Step 2</u>: Need particular sol^n $\vec{x}_p$ to

$$\vec{x}\,' = \mathbb{A}\vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

Hmmm. Try $\vec{x}_p = \vec{b}\, e^{2t}$ ← Safe! 2 is <u>not</u> an e-val

<u>Method of undetermined coefficients</u>: Plug it in. Make it work.

$$\vec{x}_p\,' = 2\vec{b}e^{2t} \overset{\text{want}}{=} \mathbb{A}\vec{b}e^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$ ← divide out e^{2t}'s

$$2\vec{b} = \mathbb{A}\vec{b} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2b_1 = -3b_1 + b_2 + 1 \\ 2b_2 = \underbrace{b_1 - 3b_2}_{\mathbb{A}\vec{b}} + 2 \end{cases}$$

$(\mathbb{A} - 2\mathbb{I})\vec{b} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{cases} -5b_1 + b_2 = -1 \\ b_1 - 5b_2 = -2 \end{cases}$$

Cramer's rule: Get $\begin{cases} b_1 = \frac{7}{24} \\ b_2 = \frac{11}{24} \end{cases}$

So $\vec{x}_p = \frac{1}{24} \begin{pmatrix} 7 \\ 11 \end{pmatrix} e^{2t}$

<u>Step 3</u>: Gen^l Sol^n $\vec{x} = \vec{x}_c + \vec{x}_p$

↑ Gen^l Sol^n to homog

↑ One stupid part sol^n

$$\underline{\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} + \frac{1}{24} \begin{pmatrix} 7 \\ 11 \end{pmatrix} e^{2t}}$$

Method Undet Coeff : $\vec{x}' = \underset{\text{const coeff}}{\mathbb{A}} \vec{x} + \underset{\text{special form}}{\vec{g}}$

$\vec{g}$	Try $\vec{x}_p =$	
$\binom{3}{4} e^{rt}$	$\vec{x}_p = \vec{b} e^{rt}$	← danger if $r =$ e.val of $\mathbb{A}$
$\binom{3}{4} t^2$	$\vec{b}_2 t^2 + \vec{b}_1 t + \vec{b}_0$	← danger if $r=0$ is e-val
$\binom{3}{4} \cos\omega t$ or $\binom{5}{6} \sin\omega t$ or combo	$\vec{A} \cos\omega t + \vec{B} \sin\omega t$	← danger if $r = \pm\omega i$ is e-val
$\binom{3}{4} t^5 e^{2t} \sin 3t$	You can guess it. Need 12 vectors!	

When in danger, use <u>variation of parameters</u>.

$$\vec{x}' = \mathbb{A}\vec{x} + \vec{g}$$

Homog sol^n : $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

Try $\vec{x}_p = u_1 \vec{x}_1 + u_2 \vec{x}_2$ where u's are fcns.

Need $\vec{x}_p\,' = \mathbb{A}\vec{x}_p + \vec{g}$

$$u_1' \vec{x}_1 + u_2' \vec{x}_2 + u_1 \vec{x}_1\,' + u_2 \vec{x}_2\,' \overset{\text{want}}{=} \underbrace{\mathbb{A}[u_1 \vec{x}_1 + u_2 \vec{x}_2]}_{u_1 \mathbb{A}\vec{x}_1 + u_2 \mathbb{A}\vec{x}_2} + \vec{g}$$

$$u_1' \vec{x}_1 + u_2' \vec{x}_2 = \vec{g}$$

$$\underbrace{\left[\vec{x}_1 \ , \ \vec{x}_2 \right]}_{X = \text{fund matrix}} \underbrace{\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}}_{\vec{u}'} = \vec{g}$$

Var of Par: $\boxed{X \vec{u}' = \vec{g}}$ 2x2 use Cramer's; nxn use Gaussian elim.

Solve for $\vec{u}'$. Integrate to get u_1, u_2.

Gen^l Sol^n: $\vec{x} = \underbrace{\left(c_1 \vec{x}_1 + c_2 \vec{x}_2 \right)}_{\vec{x}_c} + \underbrace{\left(u_1 \vec{x}_1 + u_2 \vec{x}_2 \right)}_{\vec{x}_p}$

33, 34, 35 due Wed, Proj 3 due Thurs.