Review Problems from Final Exam, Sp. '15

1. Let $y(t)$ be a solution to

$$t^2y' + 2ty = 2at, \quad \text{for } t > 0.$$

If we know that

$$y(1) = 0, y'(1) = 2,$$

?

find the constant a.

A. 2
B. 1
C. 0
D. -1
E. -2

1. Standard form: $y' + \frac{2}{t}y = \frac{2a}{t}$

2. Int. factor: $u = e^{\int P(t)\,dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

3. $\underbrace{u\left(y' + \frac{2}{t}y\right)}_{\frac{d}{dt}[uy]} = u\,\frac{2a}{t}$

$$\frac{d}{dt}\left[t^2 y\right] = t^2\cdot\frac{2a}{t} = 2at$$

So $t^2 y = \int 2at\,dt = at^2 + C \leftarrow y(1) = 0$

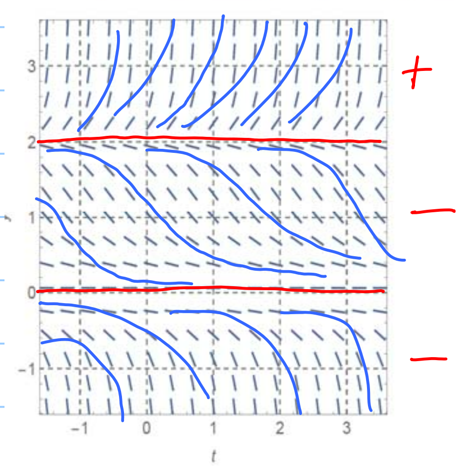
$1^2\cdot 0 = a\cdot 1^2 + C$

$\boxed{C = -a}$

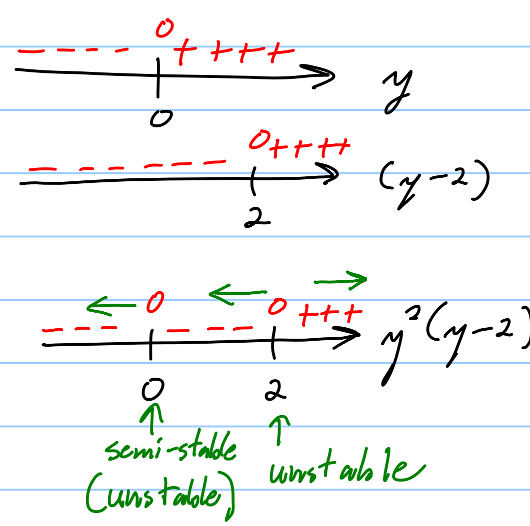
$$\boxed{y = a - \frac{a}{t^2}}$$

Use $y'(1) = 2$ now to pin down a.

2. Identify the differential equation that produces the given direction field.



A. $y' = y(y-2)$
B. $y' = y(2-y)$
C. $y' = y^2(y-2) \leftarrow$
D. $y' = y(y-2)^2$
E. $y' = y^2(y-2)^2$



3. Which of the following functions can be a solution to

$$y' = \frac{x^2+y^2}{xy}.$$

Let $v = \frac{y}{x}$.

Then $y = xv$ and

$$\boxed{y' = 1 \cdot v + x\frac{dv}{dx}}$$

A. $y = x(C - 2\ln(x))$

B. $y = x(C + 2\ln(x))$

C. $y = x\sqrt{C - 2\ln x}$

D. $y = x\sqrt{C + 2\ln x}$ ←

E. $y = Cx - 2\ln x$

$$y' = \frac{x}{y} + \frac{y}{x}$$

$$= \frac{1}{\left(\frac{y}{x}\right)} + \left(\frac{y}{x}\right) \leftarrow$$

Homogeneous !

$$\underbrace{y'}_{v + x\frac{dv}{dx}} = v + \frac{1}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v} \leftarrow \text{Separable!}$$

$$\int v\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}v^2 = \ln x + C$$

$$\frac{y}{x} \rightarrow v = \sqrt{2\ln x + 2C}$$

$$y = x\sqrt{2\ln x + \underbrace{2C}_{\tilde{C}}}$$

4. Initially, a tank contains 400 L of water with 10 kg of salt in solution. Water containing 0.1kg of salt per liter (L) is entering at a rate of 1 L/min, and the mixture is allowed to flow out of the tank at a rate of 2 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms. What is the right formulation of the differential equation for $Q(t)$?

A. $\frac{dQ}{dt} = 0.1 - \frac{2Q(t)}{400}$

B. $\frac{dQ}{dt} = \frac{0.1}{400} - \frac{2Q(t)}{400-t}$

C. $\frac{dQ}{dt} = \frac{10}{400} - \frac{2Q(t)}{400+t}$

D. $\frac{dQ}{dt} = 0.1 - \frac{2Q(t)}{400+t}$

E. $\frac{dQ}{dt} = 0.1 - \frac{2Q(t)}{400-t}$

5. Which one of the following statements is true about equilibrium solutions to $y' = 9y - y^3$?

A. $y = 3$ and $y = -3$ are unstable, $y = 0$ is asymptotically stable

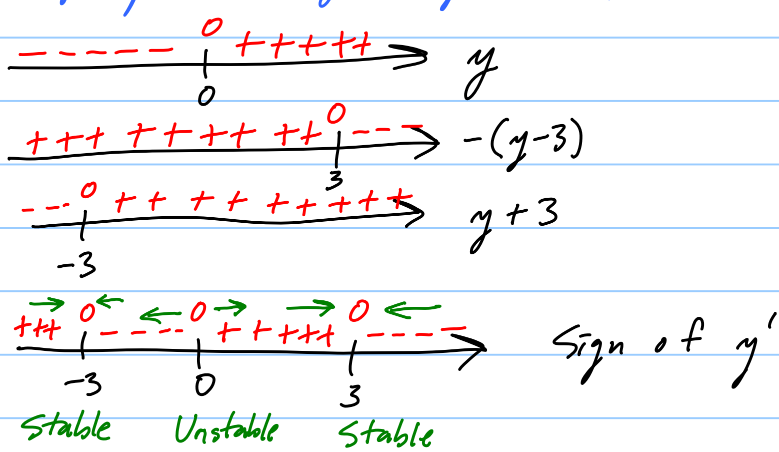
B. $y = 0, y = 3, y = -3$ are all unstable

C. $y = 0, y = 3, y = -3$ are all asymptotically stable

D. $y = 3$ and $y = 0$ are asymptotically stable and $y = -3$ is unstable

E. $y = 3$ and $y = -3$ are asymptotically stable and $y = 0$ is unstable ←

$$y' = 9y - y^3 = y(9-y^2) = y(3-y)(3+y)$$



6. The solution to the problem $2xy + 2xy^2 + 1 + (x^2 + 2x^2y + 2y)\frac{dy}{dx} = 0$ and $y(1) = 2$ is

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$$

$$\boxed{(2xy + 2xy^2+1)\,dx + (x^2+2x^2y+2y)\,dy} = 0$$

A. $y^2 + x^2 + x^2y + x = 8$

B. $y^2x^2 + x^2y + y^2 = 10$

C. $x^2y + y^2x^2 + x = 7$

D. $x^2y + y^2x^2 + y^2 + x = 11$

E. $x^2y + y^2 + x = 7$

Test for exactness: $\frac{\partial}{\partial y}\left[\text{Thing by } dx\right] \overset{?}{=} \frac{\partial}{\partial x}\left[\text{Thing by } dy\right]$

$\frac{\partial}{\partial y}(2xy+2xy^2+1) \overset{?}{=} \frac{\partial}{\partial x}(x^2+2x^2y+2y)$

$2x+4xy \overset{?}{=} 2x+4xy$

yes!

Find φ so that

(A) $\frac{\partial\varphi}{\partial x} = 2xy + 2xy^2 + 1$

(B) $\frac{\partial\varphi}{\partial y} = x^2 + 2x^2y + 2y$

Use (A): $\varphi = \int 2xy+2xy^2+1\,dx = \underline{x^2y + x^2y^2 + x + C(y)}$

Use (B): $\frac{\partial}{\partial y}\left[x^2y + x^2y^2 + x + C(y)\right] = x^2 + 2x^2y + 2y$ $\qquad \boxed{C'(y) = 2y}$

$x^2 + 2x^2y + 0 + C'(y)$ $\qquad$ So $C(y) = y^2$

Ans: $\varphi(x,y) = C$: $\quad x^2y + x^2y^2 + x + y^2 = C$ $\quad \left\{\begin{array}{l} x=1 \\ y=2 \end{array}\right.$ determine C.

7. Use the Euler method to find the approximate value at $t = 1$ for $y' = 2x + y + 1$, $y(0) = 3$ with $h = 0.5$.

x_0

A. 5

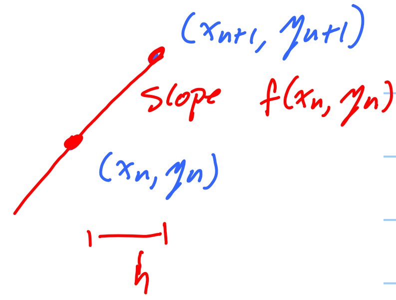
B. 8.5

C. 3

D. 6.5

E. 4.5

$$y_{n+1} = y_n + f(x_n, y_n)\cdot h$$



$x_0 = 0 \qquad y_0 = 3$

$x_1 = \frac{1}{2} \qquad y_1 = y_0 + f(x_0, y_0)h$

$= 3 + (2\cdot 0 + 3 + 1)\frac{1}{2} = 5$

$$x_2 = 1 \qquad y_2 = \underset{\uparrow \atop y_1}{5} + (2\cdot \underset{\uparrow \atop x_1}{\tfrac{1}{2}} + \underset{\uparrow \atop y_1}{5} + 1)\cdot \underset{\uparrow \atop h}{\tfrac{1}{2}}$$

8. On which interval is the initial value problem

$$\begin{cases} (5-t)y'' + (t-4)y' + 2y = \ln t \\ y(2) = 8 \end{cases}$$

guaranteed to have a unique solution?

A. $(-\infty, 5)$

B. $(0, \infty)$

C. $(0, 5)$

D. $(0, 4)$

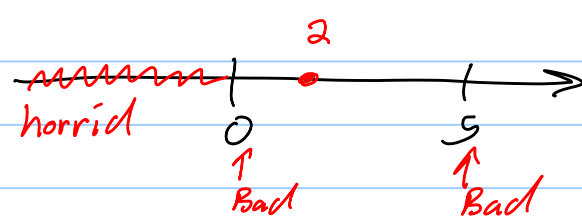
E. $(-5, \infty)$

$y(t_0) = y_0$

Step 1: Standard Form!

$$y'' + \underbrace{\frac{t-4}{5-t}}_{P} y' + \underbrace{\frac{2}{5-t}}_{Q} y = \underbrace{\frac{\ln t}{5-t}}_{R}$$

E & U Thm: Sol^n is good on largest interval about t_0 where P, Q, R are all cont.



9. Find the general solution of the differential equation

$$y'' - 10y' + 27y = 0$$

$r^2 - 10r + 27 = 0$

A. $y = c_1 e^{2t}\cos(\sqrt{5}t) + c_2 e^{2t}\sin(\sqrt{5}t)$

B. $y = c_1 e^{5t} + c_2 e^{2t}$

C. $y = c_1 e^{5t}\cos(\sqrt{2}t) + c_2 e^{5t}\sin(\sqrt{2}t)$

D. $y = c_1 e^{-5t} + c_2 e^{-2t}$

E. $y = c_1 e^{5t}\cos(\sqrt{5}t) + c_2 e^{5t}\sin(\sqrt{5}t)$

10. The function $y_1 = e^x$ is a solution of

$$(x-1)y'' - 2xy' + (x+1)y = 0.$$

If we seek a second solution $y_2 = e^x v(x)$ by reduction of order, then $v(x) =$

A. $\frac{1}{3}(x-1)^{-3}$

B. $\frac{1}{2}(x-1)^2$

C. $(x-1)^2 e^{-x}$

D. $\frac{1}{3}(x-1)^3$

E. $\frac{1}{3}(x-1)^3 e^{-x}$

Standard Form !!

$$y'' \underbrace{- \frac{2x}{x-1}} y' + \frac{x+1}{x-1} y = 0$$

$$P(x) = -\frac{2x}{x-1}$$

$$v' = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$