# Review 2

Project 3 due Thurs on Blackboard

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

Critical Points: $\begin{cases} f(x_0, y_0) = 0 \\ g(x_0, y_0) = 0 \end{cases}$

Equilibrium sol'ns: $x \equiv x_0$, $y \equiv y_0$

Huge subject: Use first order Taylor series to approximate $f$ and $g$ by linear fcns near $(x_0, y_0)$. Get linearized system. This system informs about the behavior near critical pt of actual sys.

EX: Var of Pars $\vec{x}' = A\vec{x} + \vec{g}$

$$\vec{x}' = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\vec{x} + \underbrace{\begin{pmatrix} 1 \\ e^t \end{pmatrix}}_{\vec{g}}$$

Hmmm. Undet Coeff.?
Try $\vec{x}_p = \vec{a} + \vec{b}e^t$

Step 1: Solve homog: $\vec{x}' = A\vec{x}$. Get $\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{2t}$

danger!

Var of Par to the rescue!

Step 2: Fundamental Matrix: $X = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t, \begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{2t}\right]$

$$= \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$$

$\vec{x}_p = X\vec{v}$ where $\boxed{X\vec{v}' = \vec{g}}$

$$\begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$$

For 2X2 : Cramer's Rule or use

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So $\vec{v}\,' = X^{-1}\vec{g} = \frac{1}{(-e^{3t})}\begin{bmatrix} 2e^{2t} & -3e^{2t} \\ -e^{t} & e^{t} \end{bmatrix}\begin{pmatrix} 1 \\ e^{t} \end{pmatrix}$

Get $\begin{cases} v_1' = -2e^{-t} + 3 \\ v_2' = -e^{-t} + e^{-2t} \end{cases}$

Step 3: Integrate : $v_1 = 2e^{-t} + 3t$ ← skip +C's !

$v_2 = e^{-t} - \frac{1}{2}e^{-2t}$

Step 4: $\vec{x}_p = v_1\vec{x}_1 + v_2\vec{x}_2$

$= (2e^{-t} + 3t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + (e^{-t} - \frac{1}{2}e^{-2t})\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{2t}$

Step 5: Gen^l Sol^n : $\vec{x} = \vec{x}_c + \vec{x}_p$

10. The function $y_1 = e^x$ is a solution of

$$(x-1)y'' - 2xy' + (x+1)y = 0.$$

If we seek a second solution $y_2 = e^x v(x)$ by reduction of order, then $v(x) =$

A. $\frac{1}{3}(x-1)^{-3}$
B. $\frac{1}{2}(x-1)^2$
C. $(x-1)^2 e^{-x}$
D. $\frac{1}{3}(x-1)^3$ ←
E. $\frac{1}{3}(x-1)^3 e^{-x}$

Standard Form !!

$y'' \underbrace{- \frac{2x}{x-1}}\, y' + \frac{x+1}{x-1} y = 0$

$P(x) = \frac{-2x}{x-1}$

$$\boxed{v' = \frac{1}{y_1^2} e^{-\int P(x)\,dx}}$$

$x = (x-1)+1$

$v' = \frac{1}{(e^x)^2} e^{-\int -\frac{2x}{x-1}\,dx}$

$\int \frac{2x}{x-1}\,dx = \int 2 + \frac{2}{x-1}\,dx$

$$v' = \frac{1}{e^{2x}} e^{2x + \ln(x-1)^2} = (x-1)^2$$

$$= 2x + 2\ln|x-1| \quad \text{no } +C$$
$$= 2x + \ln|x-1|^2$$

So $v = \int (x-1)^2 \, dx = \frac{1}{3}(x-1)^3 \leftarrow$ no $+C$

Or plug $y_2 = v \cdot e^x$ into ODE. Cancel out lots of things.

Get ODE in $v$: $(\quad) v'' + (\quad) v' = 0$

Let $u = v'$. First order linear in $u$: $(\quad) \frac{du}{dx} + (\quad) u = 0$. Solve for $u$.

Integrate $v' = u$

11. By the Method of Undetermined Coefficients, which of the following $Y$ is the correct form of a particular solution to the equation

$$\underbrace{y^{(4)} - 3y^{(3)} + 2y^{(2)}} = 4t - e^t + 3e^{3t}?$$

**A.** $Y = At + B + Ce^t + De^{3t}$

**B.** $Y = At^2 + Bt + Cte^t + De^{3t}$

→ **C.** $Y = At^3 + Bt^2 + Cte^t + De^{3t}$

**D.** $Y = At^3 + Bt^2 + Cte^t + Dte^{3t}$

**E.** None of the above.

$$r^4 - 3r^3 + 2r^2 = 0$$
$$r^2(r^2 - 3r + 2) = 0$$
$$r^2(r-2)(r-1) = 0$$

$r = 0, 0, 1, 2$

Homog sol$^n$: $y_c = c_1 e^{0 \cdot t} + c_2 t e^{0 \cdot t} + c_3 e^t + c_4 e^{2t}$
$$= c_1 + c_2 t + c_3 e^t + c_4 e^{2t}$$

$$y_p = \underbrace{t^2(A_1 t + A_0)}_{y_{p_1}} + \underbrace{tBe^t}_{y_{p_2}} + \underbrace{Ce^{3t}}_{y_{p_3}}$$

12. The general solution to the homogeneous equation $t^2y'' + 7ty' + 5y = 0$ on the interval $0 < t < \infty$ is $y_c(t) = c_1t^{-1} + c_2t^{-5}$. A particular solution to the inhomogeneous equation $t^2y'' + 7ty' + 5y = t$ has the form $y_p(t) = u_1(t)t^{-1} + u_2(t)t^{-5}$. Which of the following are satisfied by $u_1'$ and $u_2'$?

A. $u_1' = -t^5,\ u_2' = t$

→ B. $u_1' = \frac{t}{4},\ u_2' = -\frac{t^5}{4}$

C. $u_1' = \frac{t^3}{4},\ u_2' = -\frac{t^7}{4}$

D. $u_1' = -t^7,\ u_2' = t^3$

E. $u_1' = t,\ u_2' = t^5$

Undet coeff only works if left hand side is const coeff linear! Var of Par

Standard Form!

$$y'' + \frac{7}{t}y' + \frac{5}{t^2} = \underbrace{\boxed{\frac{1}{t}}}_{F}$$

Homog sol$^n$ $c_1 \underbrace{t^{-1}}_{y_1} + c_2 \underbrace{t^{-5}}_{y_2}$

$y_p = u_1 y_1 + u_2 y_2$

where $u_1' = \frac{-y_2 F}{W}$ $\quad u_2' = \frac{y_1 F}{W}$

$$W = \det \begin{bmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{bmatrix} = -4t^{-7}$$

$$\begin{cases} u_1' = \dfrac{-(t^{-5})\left(\frac{1}{t}\right)}{-4t^{-7}} = \dfrac{1}{4}t \\[2ex] u_2' = \dfrac{(t^{-1})\left(\frac{1}{t}\right)}{-4t^{-7}} = -\dfrac{1}{4}t^5 \end{cases}$$

Systems: Let $x_1 = y$
$x_2 = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = y'' = \text{something from ODE} \end{cases}$$

Var of Pars for systems yields same answer!

13. An undamped, free vibration $u'' + 9u = 0$ has initial conditions $u(0) = 4, u'(0) = 9$. The solution of this initial value problem can be written as $u = R\cos(\omega t - \delta)$. What are $R$ and $\delta$?

**A.** $R = 5$, $\delta = \frac{\pi}{2}$

**B.** $R = 6$, $\delta = \frac{\pi}{2}$

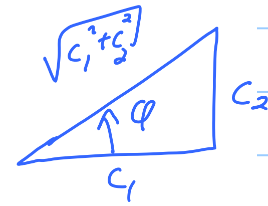
→ **C.** $R = 5$, $\delta = \tan^{-1}(\frac{3}{4})$

**D.** $R = 2\sqrt{5}$, $\delta = \tan^{-1}(\frac{3}{4})$

**E.** $R = 2\sqrt{5}$, $\delta = \tan^{1}(\frac{1}{2})$

$r = \pm 3i$

$$u = c_1 \cos 3t + c_2 \sin 3t$$
$$= R\cos(3t - \varphi)$$




$e^{i\alpha}$ $e^{i\beta}$

$$R = \sqrt{c_1^2 + c_2^2}$$

want

$$u(0) = c_1 \cdot 1 + c_2 \cdot 0 = 4$$
$$u'(0) = -3c_1 \cdot 0 + 3c_2 \cdot 1 = 9$$

$$\boxed{\begin{aligned} c_1 &= 4 \\ c_2 &= 3 \end{aligned}}$$

$$R = \sqrt{3^2 + 4^2} = 5$$

$$\varphi = \tan^{-1}\left(\frac{3}{4}\right)$$