Review 3

Final Exam, Thurs, May 3, 1:00-3:00 pm in Elliott Hall of Music – 20 multiple choice questions
Last page = Laplace Transform Table

Best kind of gen^l sol^n to homog prob:

1) 2^{nd} order linear homog eqn: $y = c_1 y_1 + c_2 y_2$

with $\begin{cases} y_1(0) = 1 \\ y_1'(0) = 0 \end{cases}$ $\begin{cases} y_2(0) = 0 \\ y_2'(0) = 1 \end{cases}$ $W(0) = 1$ ✓

Normalized Fund sol^n

EX: $y'' + y = 0$ $y = c_1 \cos x + c_2 \sin x$

EX: $y'' - y = 0$ $y = c_1 e^{x} + c_2 e^{-x} = \tilde{c}_1 \cosh x + \tilde{c}_2 \sinh x$

2) $\vec{x}' = A\vec{x}$. $\vec{x}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{x}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Normalized Fund. Matrix $X = [\vec{x}_1, \vec{x}_2]$. Then $X(0) = I$.

14. Let $F(s) = \dfrac{6}{(s-3)^3}$ and $G(s) = \dfrac{5}{s^2+25}$. What is the inverse Laplace transform of the product of $F(s)G(s)$?

A. $3\int_0^t (t^2-\tau)e^{3t-\tau}\sin(5\tau)\,d\tau$

B. $6\int_0^t (t^2-\tau)e^{3(t-\tau)}\sin(5\tau)\,d\tau$

→ C. $3\int_0^t (t-\tau)^2 e^{3(t-\tau)}\sin(5\tau)\,d\tau$

D. $6\int_0^t (t-\tau)^2 e^{3(t-\tau)}\sin(5\tau)\,d\tau$

E. $3\int_0^t (t-\tau)^2 e^{3(t-\tau)}\sin(5(t-\tau))\,d\tau$

$\mathcal{L}[t^2] = \dfrac{2!}{s^3}$

$\mathcal{L}[e^{at} f(t)] = F(s-a)$

So $\mathcal{L}[\underbrace{3e^{3t} t^2}_{f(t)}] = \dfrac{2 \cdot 3}{(s-3)^3}$

$\mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2}$

So $g(t) = \sin 5t$

$\mathcal{L}[f * g] = F(s)\,G(s)$

Ans: $(f*g)(t) = (g*f)(t)$

$\int_0^t g(\tau)\, f(t-\tau)\, d\tau$

$\int_0^t \sin(5\tau)\left[3e^{3(t-\tau)}(t-\tau)^2\right] d\tau$

$\boxed{\begin{array}{l} y'' + y = g(t) \quad y(0) = 3,\ y'(0) = 4. \\ \text{Get formula for sol}^n. \end{array}}$

Note: $f*g$ for the Fourier Transform is different.

15. Which of the following functions has Laplace transform equal to

$$\frac{3s^2+4s-1}{(s+1)(s^2+2s+5)}? = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

A. $-\frac{1}{2}e^{-t} + \frac{7}{2}\cos(2t) + \frac{3}{4}\sin(2t)$

B. $-\frac{1}{2}e^{-t} + \frac{7}{2}e^{-t}\cos(2t) - e^{-t}\sin(2t)$

C. $-\frac{1}{2}e^{-t} + \frac{7}{2}e^{-t}\cos(2t) + \frac{3}{4}e^{-t}\sin(2t)$

D. $3e^{-t}\cos(2t) - e^{-t}\sin(2t)$

E. $-\frac{1}{2}e^{-t} + 3e^{-t}\cos(2t) - e^{-t}\sin(2t)$

$$3s^2+4s-1 = A(s^2+2s+5) + (Bs+C)(s+1)$$

$$= \underbrace{[A+B]}_{3}s^2 + \underbrace{[2A+B+C]}_{4}s + \underbrace{[5A+C]}_{-1}$$

$$\frac{Bs+C}{s^2+2s+5} = \frac{Bs+C}{(s+1)^2+2^2}$$

$$\mathcal{L}^{-1}\left[\frac{(s-a)}{(s-a)^2+b^2}\right] = e^{at}\cos bt$$

$$= \frac{B[(s+1)-1]+C}{(s+1)^2+2^2} = B\cdot\underbrace{\frac{(s+1)}{(s+1)^2+2^2}}_{\mathcal{L}[e^{-t}\cos 2t]} + \frac{(C-B)}{2}\cdot\underbrace{\frac{2}{(s+1)^2+2^2}}_{\mathcal{L}[e^{-t}\sin 2t]}$$

P.F. $$\frac{s^7+s+1}{(s-2)^4(s^2+2s+5)^2} = \underbrace{\frac{A_1}{(s-2)^4}+\cdots+\frac{A_4}{(s-2)}}_{} + \underbrace{\frac{Bs+C}{s^2+2s+5} + \frac{Ds+E}{(s^2+2s+5)^2}}_{}$$

$[(s+1)-1]$

$$\frac{s}{(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1} = \frac{1}{(s+1)} - \frac{1}{(s+1)^2}$$

16. Solve the initial value problem

$$\mathcal{L}: \quad y'' + 4y' + 5y = \delta(t-\frac{\pi}{4}), \quad y(0)=1, \; y'(0)=-2.$$

$\delta(t)$ ← Boom at $t=0$

$\delta(t-\frac{\pi}{4}) = \delta_{\pi/4}(t)$ ← Boom at $t=\frac{\pi}{4}$

A. $e^{-2t}\cos t + u_{\frac{\pi}{4}}(t)e^{\frac{\pi}{2}-2t}\sin(t-\frac{\pi}{4})$

B. $e^{-2t}\cos t - u_{\frac{\pi}{4}}(t)e^{\frac{\pi}{2}-2t}\sin(t-\frac{\pi}{4})$

C. $e^{-2t}\cos t + u_{\frac{\pi}{4}}(t)e^{2t}\sin t$

D. $e^{2t}\cos t - u_{\frac{\pi}{4}}(t)e^{-2t}\sin t$

E. $e^{2t}\cos t + u_{\frac{\pi}{4}}(t)e^{-2t}\sin t$

$$\mathcal{L}[\delta_a(t)] = e^{-as} \qquad \mathcal{L}[\delta(t)] = 1$$

$$\int_0^\infty \delta_a(t)f(t)\,dt = f(a)$$

3

$$\left[s^2 Y - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{-2}\right] + 4\left[sY - \underbrace{y(0)}_{1}\right] + 5Y = e^{-\pi/4\, s}$$

Get $\quad Y = H(s) + e^{-\pi/4\, s} \cdot \underbrace{\dfrac{1}{s^2+4s+5}}_{G(s)}$

$\uparrow$ Rational

$$\mathcal{L}\left[u_c(t) f(t-c)\right] = e^{-cs} F(s)$$

17. The solution $x_2(t)$ determined by the initial value problem

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$ $\qquad r^2 - 1 = 0$, $r = \pm 1$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1 \end{cases} \leftarrow \quad x_2 = x_1'$$

$(x_1')' = x_1$

$x_1'' - x_1 = 0$

$$\begin{cases} x_1 = c_1 e^{t} + c_2 e^{-t} \\ x_2 = x_1' = c_1 e^{t} - c_2 e^{-t} \end{cases}$$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$ is given by

A. $x_2(t) = 2e^{-t} + 3e^{t}$

B. $x_2(t) = 2\sin t + \cos t$

C. $x_2(t) = \frac{1}{2}e^{-t} + \frac{3}{2}e^{t}$

D. $x_2(t) = 2\cos t - \sin t$

E. None of the above

$\det \begin{bmatrix} 0-r & 1 \\ 1 & 0-r \end{bmatrix} = 0$

$r^2 - 1 = 0$

$r = \pm 1$

For $r=1$: Get $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $r=-1$: $(A - (-1)I)\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ 1 & 0-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} a_1 + a_2 = 0 \\ a_1 = -a_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a_2 free. Let $a_2 = c$.

$a_1 = -c$

$\vec{a} = \begin{pmatrix} -c \\ c \end{pmatrix} = c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Take $c=-1$: Get $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

18. In the phase portrait of the system

det $\begin{bmatrix} -3-r & -5 \\ 3 & 1-r \end{bmatrix}$

$$\mathbf{x}' = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} \mathbf{x},$$

the origin is a

A. asymptotically stable spiral point
B. saddle point
C. asymptotically stable node
D. asymptotically unstable node
E. asymptotically unstable spiral point

$r_1 < r_2 < 0$ $\quad e^{-t\text{'s}}$

A. Stable Improper Node

$0 < r_1 < r_2$ $\quad e^{+t\text{'s}}$

Unstable

$r = a \pm bi$

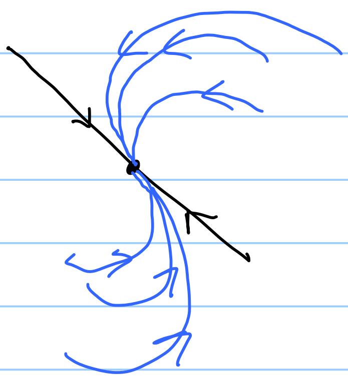
e^{at} (Cos and Sin bt)

$a < 0$ Asym. Stable Spiral

$a > 0$ Unstable

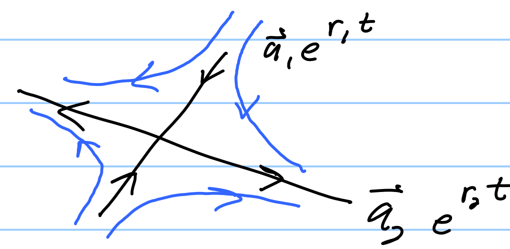
Repeated roots:

$r_1 = r_2 < 0$



$r_1 < 0 < r_2$

Unstable Saddle Point



19. Find the solution of the following initial value problem

$$\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}. \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A. $\mathbf{x}(t) = 4 \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$

B. $\mathbf{x}(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t}$

C. $\mathbf{x}(t) = -\frac{5}{7} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \frac{2}{7} \begin{pmatrix} 6 \\ 1 \end{pmatrix} e^{4t}$

D. $\mathbf{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 6 \\ 1 \end{pmatrix} e^{4t}$

E. $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 6 \\ 1 \end{pmatrix} e^{4t}$

Office Hours next week
M, T, W 2-3 pm