MA 341 Foundations of real analysis (Advanced calculus)

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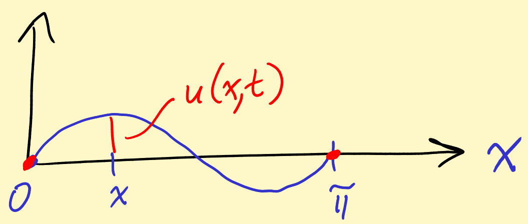
HWK 1 on Home page due Wed. 1/17 in Gradescope

Book: 4th edition Bartle, Sherbert

Bernoulli vs Bernoulli: Vibrating string problem.

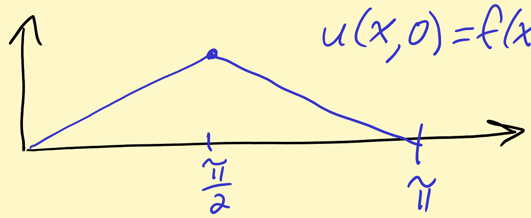


Harpsichord prob

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Bound cond: $\begin{cases} u(0,t)=0 \\ u(\pi,t)=0 \end{cases}$

Initial cond:



$u(x,0)=f(x)$ ← given

Harpsichord plucked: $\frac{\partial u}{\partial t}(x,0) \equiv 0.$

Jacob's solⁿ: $u(x,t) = \frac{1}{2}\left[f(x+ct)+f(x-ct)\right]$

Johann: Fourier series. Try to find solⁿs

$$\boxed{u(x,t) = X(x)T(t)}$$ [Separation of vars]

Let $c=1$.

Plug into: $u_{tt} = u_{xx}$

$$X T'' = X'' T$$

$$\lambda = \frac{T''}{T} = \frac{X''}{X}$$

Let $x = \frac{\pi}{2}$. See LHS = const. Let $t = 3$. See RHS same const.

$$\boxed{X'' - \lambda X = 0 \qquad T'' - \lambda T = 0}$$

B.C. $\begin{cases} u(0,t) = X(0)\,T(t) \overset{\text{want}}{=} 0 \\ u(\pi,t) = X(\pi)\,T(t) = 0 \end{cases}$

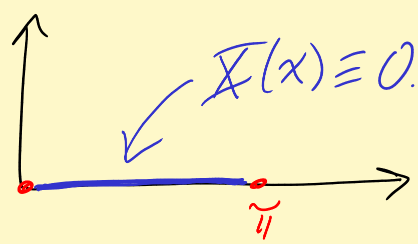
Don't want $T(t) \equiv 0$. zero sol^n ✓

Need $X(0) = 0$, $X(\pi) = 0$

Sturm-Liouville prob: $\boxed{\begin{array}{l} X'' - \lambda X = 0 \\ \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \end{array}}$

Case $\lambda = 0$: $X'' \equiv 0$.

$$X(x) = \underbrace{Ax + B}_{\text{line}}$$



Only get zero sol^n

Case $\lambda > 0$: $\lambda = k^2$. $\quad X'' - k^2 X = 0$

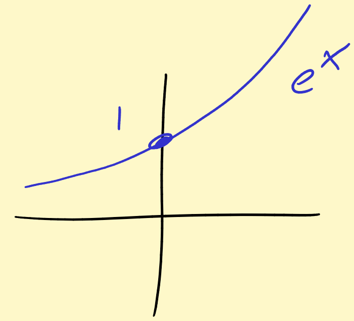
$$r^2 - k^2 = 0, \quad r = \pm k.$$

Sol^n $\quad X(x) = c_1 e^{kx} + c_2 e^{-kx}$

$X(0) = c_1 + c_2 \overset{\text{want}}{=} 0 \qquad$ So $\boxed{c_2 = -c_1}$

$$X(x) = c_1\left(e^{kx} - e^{-kx}\right)$$

$$X(\pi) = c_1\Big(\underbrace{\underset{>1}{e^{k\pi}} - \underset{<1}{e^{-k\pi}}}_{\text{not zero}}\Big)$$



Must have $c_1 = 0$. $c_2 = -c_1 = 0$. Ouch! zero sol^n

Case $\lambda < 0$: $\lambda = -k^2$ $\quad X'' + k^2 X = 0$

$$r^2 + k^2 = 0$$

$$r = \pm ki$$

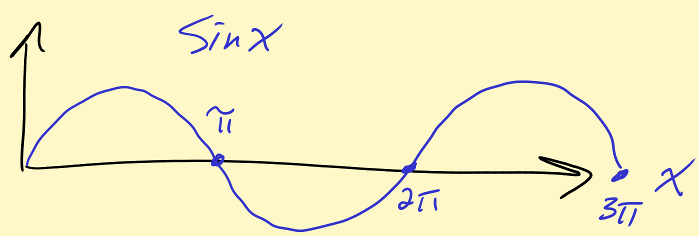
$$X(x) = c_1 \sin kx + c_2 \cos kx$$

$$X(0) = c_1 \cdot 0 + c_2 \cdot 1 \overset{\text{want}}{=} 0 \qquad \boxed{c_2 = 0}$$

$$X(x) = c_1 \sin kx$$

$$X(\pi) = c_1 \sin k\pi$$ Don't want $c_1 = 0$ too.

Need $\sin k\pi = 0$



Need $k\pi = \pi, 2\pi, 3\pi, 4\pi, \ldots, n\pi, \ldots$

Need $k = 1, 2, 3, \ldots$ $\qquad k \in \mathbb{N}$

Get bunch of λ's : $\lambda = -n^2$, $n \in \mathbb{N}$.

Back to T prob: $T'' - (-n^2)T = 0$.

$$T(t) = \underbrace{A \sin nt}_{t=0.\ \text{Velocity of pluck.} =0.} + \underbrace{B \cos nt}_{t=0.\ \text{Shape}}$$

Take $A=0$.

Johann: $u_n(x,t) = \sin nx \cos nt$

Initial cond? Big idea: Try $u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt$

want

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = f(x)$$

Key. $\sin nx$ are $\perp$ on $[0,\pi]$:

$$\underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{\langle \sin nx,\ \sin mx \rangle} = \begin{cases} \frac{\pi}{2} & n=m \\ 0 & n \neq m \end{cases}$$

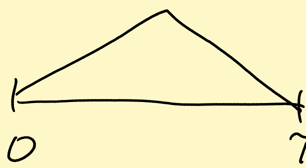
$$\int_0^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{\substack{=0 \quad n\neq m \\ =\frac{\pi}{2} \quad n=m}}$$

$$= \frac{\pi}{2} A_m$$

A_m has to be $\quad \frac{2}{\pi}\int_0^{\pi} f(x)\sin mx\,dx$

Pluke shape: $A_1 = \frac{1}{1^2}$, $A_2 = 0$, $A_3 = \frac{-1}{3^2}$, $A_4 = 0$, $A_5 = \frac{1}{5^2}$, ...

$$u(x,0) = \frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots$$

Does that converge to 

? Yes