Lesson 2 Math induction (1.2)

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EX: $1+2+\cdots+n = \frac{n(n+1)}{2}$ $P(n)$ = statement about n

$P(1)$ is true: $P(1)$: $1 = \frac{1(1+1)}{2}$ ✓

Suppose we know that $P(N)$ true. If we can deduce $P(N+1)$ from $P(N)$, then $P(n)$ true for all n.

[Princ of Math Induction: PMI]

$P(N)$: $1+2+\cdots+N = \frac{N(N+1)}{2}$ ← assume we know

$P(N+1)$: $\underbrace{1+2+\cdots+N}_{\frac{N(N+1)}{2}} + (N+1) = \frac{N(N+1)+2(N+1)}{2}$

$= \frac{(N+1)[N+2]}{2}$ ← $(N+1)-1 = N$

$= \frac{(N+1)[(N+1)+1]}{2}$ ✓

Believe the PMI:

$P(1)$ ✓

$P(1) \Rightarrow P(2)$ ✓

$P(2) \Rightarrow P(3)$ ✓

$\vdots$

EX Show that $n < 2^n$ for all n.

$n=1$: $1 < 2^1$ ✓

Induction hyp. Suppose we know true for N:

$$N < 2^N$$

Need to deduce that $(N+1) < 2^{(N+1)}$ from this

Hmmm $\underbrace{N}_{<2^N} + 1 < 2^N + \underbrace{1}_{\le 2^N}$ So $N+1 < \underbrace{2^N + 2^N}_{2 \cdot 2^N = 2^{N+1}}$ ✓

<u>EX</u> $5^{2n} - 1$ is divisible by 8 for $n \in \mathbb{N}$.

$n = 1$: $5^2 - 1 = 24$ ✓

Suppose we know $P(N)$ true: $5^{2N} - 1$ div by 8.

Want to see $P(N+1)$: $\underbrace{5^{2(N+1)}}_{5^{2N+2} = 25 \cdot 5^{2N}} - 1$ div by 8?

$$= 25\left(5^{2N} - 1 + 1\right) - 1$$

$$= 25\underbrace{\left(5^{2N} - 1\right)}_{\text{div by 8}} + \underbrace{24}_{\text{div by 8}} \quad \checkmark \text{ div by 8}$$

Hint for 1.2:10

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \underbrace{\frac{1}{(2n-1)(2n+1)}}_{\text{Partial fractions!}} = ?$$

$$\frac{A}{2n-1} + \frac{B}{2n+1}$$

$$\frac{1/2}{2n-1} - \frac{1/2}{2n+1}$$

$$\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

Telescoping series! HWK: verify using PMI.

<u>EX</u>: All cows are same color. n cows

$P(1)$ true.

Suppose $P(N)$ true.

$P(N+1)$? Of course!

Mistake: $P(1)$ does <u>not</u> imply $P(2)$. PMI crashes!

Peano postulates. Last one PMI.

<u>Well ordering property of $\mathbb{N}$ (WOP)</u>

Every non-empty subset of $\mathbb{N}$ has a <u>least</u> element.

Claim: <u>WOP $\Longrightarrow$ PMI</u>

why: $P(n)$: Statement about natural number n.

[We know $P(1)$ true.
We know that if $P(N)$ true, then $P(N+1)$ true.

We know WOP, but not PMI.

Let $S = \{n : P(n) \text{ is true}\}$. Want to see $S = \mathbb{N}$.

Suppose $S \neq \mathbb{N}$. Then $\mathbb{N} \setminus S$ is non-empty.

WOP gives <u>least</u> element $n_0 \in \mathbb{N} \setminus S$.

HMMM. $1 \in S$. So $n_0 \neq 1$. $n_0 > 1$.

$\underbrace{n_0 - 1}_{\text{in } S!} < n_0 \leftarrow$ least one in $\mathbb{N} \setminus S$.

Aha! Induction hyp yields

$$P(n_0-1) \implies P(n_0) \text{ true} \quad ↯ \quad \underline{\text{contradiction}}$$

Assumption that $S \neq \mathbb{N}$ must be wrong. So $S = \mathbb{N}$ ✓

Words from 1.1

injective : one-to-one
surjective : onto
bijective : one-to-one and onto

$$\underbrace{A \setminus B} = \{x : x \in A \ \underline{\text{and}}\ x \notin B\}$$

$A - B$ same thing

HWK 1 due in Gradescope (find link in Brightspace)