Lesson 3 infinite vs INFINITE! (1.3)

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{N}} > \sqrt{N}$$ Piazza

Def^n Set S is finite if $\exists\ \varphi : \{1,2,3,\ldots,n\} \xrightarrow[\text{onto}]{1-1} S$.

Note : n is uniquely determined. # elements S (Cardinality).

Def^n S is called infinite if it is not finite.

[No $\varphi : S \xrightarrow[\text{onto}]{1-1} \{1,\ldots,n\}$ for any n.]

Big question : Are some infinite sets bigger than others?

EX $\mathbb{N}$ and even numbers $\{2,4,6,\ldots\}$
$\{2\cdot1, 2\cdot2, 2\cdot3, \ldots\}$

Aha! $\varphi(n) = 2n$ is a bijection $\mathbb{N} \longrightarrow \{\text{Even } n\}$

Think: These 2 infinite sets are the same "size" (cardinality).

Words 1) S is denumarable if it is infinite and $\exists\ \varphi : \mathbb{N} \xrightarrow[\text{onto}]{1-1} S$.

2) S is countable if S is finite or denumerable.

Next $\mathbb{N} \times \mathbb{N} = \{ (n,m) : n, m \in \mathbb{N} \}$

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
(4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
(3,1)\,6 & (3,2) & (3,3) & (3,4) & \cdots \\
(2,1)\,2 & (2,2)\,5 & (2,3) & (2,4) & \cdots \\
1\,(1,1) & (1,2)\,3 & (1,3)\,4 & (1,4) & \cdots
\end{array}
$$

Aha! $\varphi : \underset{\substack{\uparrow \\ \text{size} \\ \infty}}{\mathbb{N}} \xrightarrow[\text{onto}]{1\text{-}1} \underbrace{\mathbb{N} \times \mathbb{N}}_{\substack{\uparrow \\ \text{size "}\infty \times \infty\text{" !}}}$

EX: $\mathbb{Q}^+$

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
\frac{1}{3} & \frac{2}{3} & \cancel{\frac{3}{3}} & \frac{4}{3} & \cdots \\
\cancel{\frac{1}{2}} & \cancel{\frac{2}{2}} & \frac{3}{2} & \underbrace{\frac{4}{2}}_{2} & \cdots \\
\underbrace{\frac{1}{1}}_{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots
\end{array}
$$

Remove crossed out ones.
Slide over.
Count with the zig-zag method.

See $\mathbb{Q}^+$ is countable!

How to show $\mathbb{Q}$ and $\mathbb{N}$ have same cardinality.

$$
\mathbb{N} \begin{cases} \text{Even } \mathbb{N} : \xleftarrow{2n} \!\!\rightarrow \mathbb{N} \longleftrightarrow \mathbb{Q}^+ \\ \text{Odd } \mathbb{N} : \xleftarrow{2n-1} \!\!\rightarrow \mathbb{N} \longleftrightarrow \mathbb{Q}^- \end{cases}
$$

$$\mathbb{N} \longleftrightarrow \mathbb{Q}^+ \cup \mathbb{Q}^-$$

Hmmm. $\varphi(n) = 2n+1$ $\quad \varphi : \mathbb{N} \longrightarrow \{3, 5, 7, \cdots\}$

$$\mathbb{N} \left\{ \begin{array}{ccc} (\text{EVEN}) & \longleftrightarrow & \mathbb{Q}^+ \\ (\text{ODD} \geq 3) & \longleftrightarrow & \mathbb{Q}^- \\ \{1\} & \longleftrightarrow & \{0\} \end{array} \right\} \quad \mathbb{Q}$$

Tricky little arguments show

<u>Thm</u> Suppose S_1, S_2 sets and $S_1 \subset S_2$.

a) If S_2 is <u>countable</u>, so is S_1.

b) If S_1 is <u>uncountable</u>, so is S_2.

<u>Thm</u> The following are equivalent

1) S is <u>countable</u>

2) $\exists\ \varphi : \mathbb{N} \xrightarrow{\text{onto}} S$

3) $\exists\ \psi : S \xrightarrow{1-1} \mathbb{N}$

<u>Thm</u> Countable union of countable sets is countable.

<u>Sketch of pf</u> :

$$A_1 = \{a_{11}, a_{12}, a_{13}, \dots\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \dots\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \dots\}$$

Georg Cantor $(0,1)$ is "bigger" than $\mathbb{N}$.

r_1 : $.d_{11}\, d_{12}\, d_{13}\, d_{14} \cdots$

r_2 : $.d_{21}\, d_{22}\, d_{23}\, d_{24} \cdots$

r_3 : $.d_{31}\, d_{32}\, d_{33}\, d_{34} \cdots$

$\vdots$ $\vdots$

"Diagonal" argument : $D = .D_1\, D_2\, D_3 \cdots$

D_1 different than d_{11}

D_2 different than d_{22}

D_3 different than d_{33}

$\vdots$

D does not get hit by $\varphi(n) = r_n$!

$\mathbb{R} \times \mathbb{R}$ same size as $\mathbb{R}$! Like $\mathbb{N} \times \mathbb{N} \longleftrightarrow \mathbb{N}$.

<u>Cantor</u> : Set A. $P(A)$ = "power set of A"

= Set of all subsets of A.

Note: $\emptyset \in P(A)$, $A \in P(A)$.

Cantor's Thm $P(A)$ is "bigger" than A.

There is no $\varphi : A \xrightarrow[\text{onto}]{1\text{-}1} P(A)$.

In fact, there is no $\varphi : A \xrightarrow{\text{onto}} P(A)$.

Pf: Suppose $\varphi : A \xrightarrow{\text{onto}} P(A)$.

$$D = \{ a \in A : a \notin \varphi(a) \}$$

Note: D is a subset of A. So $D \in P(A)$.

φ onto. So $\exists\ a_0 \in A$ with $\varphi(a_0) = D$.

Hmmm. Is $a_0 \in D$ or $a_0 \notin D$?

Fact: $a_0 \in D$ or $a_0 \notin D$.

Case $a_0 \in D$: $a_0 \notin \underbrace{\varphi(a_0)}_{D}$ $\lightning$

Case $a_0 \notin D$: Not case that $a_0 \notin \underbrace{\varphi(a_0)}_{D}$ $\lightning$

Contradiction! means that no such φ exists.