Lesson 4 Basic properties of $\mathbb{R}$. Read Chap 2. (Today: 2.1)

HWK 1 due by 11:30 pm in Gradescope tonight.
HWK 2 on Home page due next Wed.

Remarks If $S = \{1, 2, \ldots, n\}$. The "cardinality" of S is n. $P(S)$ = set of all subsets of S.

Card of $P(S) = 2^n$ ← Explains "power set"

Card of $\mathbb{N}$ is $\aleph_0$ = "aleph naught"

Card of $P(S)$ is $2^{\aleph_0}$.

Card of $\mathbb{R}$ is c ← "Continuum"

Continuum hypothesis: $c = 2^{\aleph_0}$. No Card between $\aleph_0$ and c.

Richard Dedekind: "Was sind und was sollen die Zahlen?"

$$1 \sim \emptyset, \quad 2 \sim \{\emptyset\}, \quad 3 \sim \{\{\emptyset\}\}, \ldots$$

Similar to Peno's work, but Dedekind had a model for $\mathbb{N}$.

Mathematical logic. Gödel's Incompleteness Thm.

Axioms for $\mathbb{R}$. Field axioms p. 24. (2.1)

Baby, dumb sounding thms on p. 24-25: Read carefully.

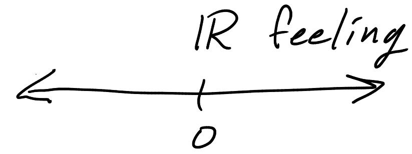
[Want a small set of axioms.]

Gödel: There are always facts about $\mathbb{N}$ that are <u>true</u>, but <u>not provable</u> via "the rules".

P = unprovable statement.

Can add P <u>or</u> not P to the axioms!

From $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ to $\mathbb{R}$.



Greeks: $\sqrt{2}$ is irrational.

<u>Pf</u>: Suppose $2 = \left(\frac{p}{q}\right)^2$ $\quad p, q$ "relatively prime"

meaning they have no common factors.

$$p = p_1^{n_1} p_2^{n_2} \cdots p_N^{n_N} \qquad (2, 3, 5, 7, 11, \ldots)$$

$$q = q_1^{m_1} q_2^{m_2} \cdots q_M^{m_M}$$

$$\left[\frac{p}{q} = \cdots \frac{p_K^{n_K}}{(p_K)^{m_\ell}} \cdots \qquad \begin{array}{l} m_\ell > n_K: \text{ term } \frac{1}{p_K^{m_\ell - n_K}} \\ m_\ell < n_K: \text{ term } \frac{p_K^{n_K - m_\ell}}{1} \end{array} \right]$$

$$q_\ell = p_K$$

$$2 = \frac{p^2}{q^2} \qquad 2q^2 = p^2$$

$$2 \cdot q_1^{2m_1} q_2^{2m_2} \cdots q_M^{2m_M} = p_1^{2n_1} \cdot p_2^{2n_2} \cdots p_N^{2n_N}$$

p_1 must equal 2. $n_1 \geq 1$. 2^2 divides RHS.

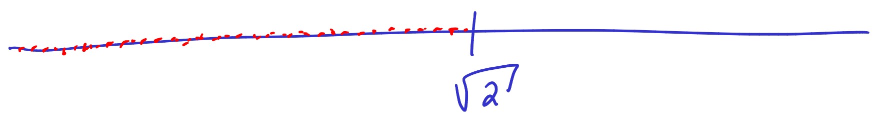
2 must divide product of powers of q's.

So q_1 must also be equal to 2. But p and q have no common factors! $\lightning$. No such $p/q \in \mathbb{Q}$.

Dedekind cuts (schnitts)

Identify real number r with $\{q \in \mathbb{Q} : q \le r\}$.

$$\sqrt{2} \sim \{q \in \mathbb{Q}^+ : q^2 < 2\} \cup \{q \in \mathbb{Q} : q \le 0\}$$



Order properties of $\mathbb{R}$

$\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$.

Book: $\mathbb{R}^+ = \mathbb{P}$

1) If $a, b \in \mathbb{R}^+$, then so is $a+b$.
2) If $a, b \in \mathbb{R}^+$, then so is ab
3) If $a \in \mathbb{R}$, then exactly one of

$$\underbrace{a \in \mathbb{R}^+, \quad a = 0, \quad \text{or } -a \in \mathbb{R}^+}_{\text{Trichotomy property.}}$$

Def^n $a, b \in \mathbb{R}$. $a < b$ (same as $b > a$) means $b - a \in \mathbb{R}^+$.

$a \le b$ (same as $b \ge a$) means $b - a \in \mathbb{R}^+ \cup \{0\}$.

Read thms on p. 27 and their proofs.

Thm: If $a \in \mathbb{R}$ and $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then a must be 0.

[No smallest positive real.]

Fun thing: $a, b \in \mathbb{R}^+$

$$\underbrace{(ab)^{1/2}}_{\text{Geom mean}} \leq \underbrace{\frac{a+b}{2}}_{\text{Arithmetic mean}}$$

Case $a=b$. Duh. ✓

Case $a \neq b$.

Then $\sqrt{a} \neq \sqrt{b}$. ← otherwise square. See $a=b$. ↯.

$$\left(\sqrt{a}-\sqrt{b}\right)^2 > 0$$

$$a - 2\sqrt{a}\sqrt{b} + b > 0$$

$$a+b > 2\sqrt{ab} \quad ✓$$

Fact: Only get equality in $\sqrt{ab} \leq \frac{a+b}{2}$ when $a=b$.

Fancier version: $$(a_1 \cdot a_2 \cdots a_n)^{1/n} \leq \frac{a_1+a_2+\cdots+a_n}{n}$$

Pf: $\ln x$ is a convex fcn $(x>0)$.