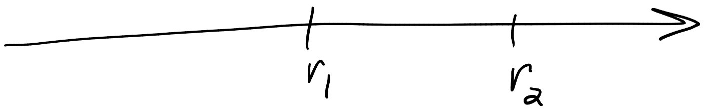
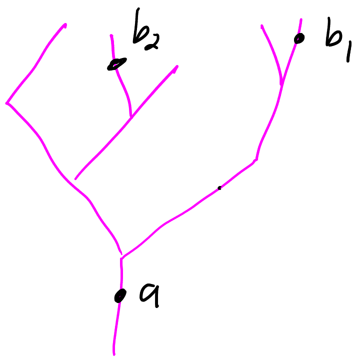
Lesson 5 Order, distance, topology on $\mathbb{R}$ (2.2)



$\mathbb{R}$ is well-ordered: Given $r_1 \neq r_2$. Then $r_1 < r_2$ or $r_2 < r_1$. [Trichotomy principle]

EX



$a < b_1$
$a < b_2$
not related

Not well-ordered.
(partially ordered)

2.2 $|a|$ absolute value

Think: $\text{dist}(r_1, r_2) = |r_1 - r_2|$

Def^n:

$$|a| = \begin{cases} a & a > 0 \\ 0 & a = 0 \\ -a & a < 0 \end{cases}$$

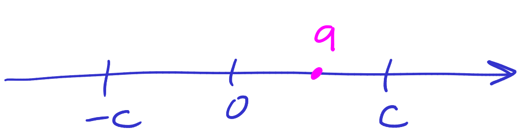
$\leftarrow$ $\text{dist}(a, 0) = -a$

Facts

1) $|ab| = |a|\,|b|$

2) $|a^2| = |a|\,|a| = |a|^2 = \left[\begin{cases} a^2 \\ 0^2 \\ (-a)^2 = a^2 \end{cases} \right] = a^2$

3) If $c > 0$, then $|a| \le c$ if and only if $-c \le a \le c$.



In set notation $\{a : |a| \le c\} = [-c, c]$.

Read book carefully: Ex: in pf of (3):

If $|a| \le c$, then we have both $a \le c$ and $-a \le c$ (why?).

Don't skip the (why?)s: Case $a \ge 0$: $\underbrace{|a|}_{=a} \le c$ ✓

$-a \le 0 \le c$ ✓

Case $a < 0$: similar

EX: Baby fact: $(-1)a = -a$

$$0 = 0 \cdot a = [1 + (-1)] \cdot a = \underbrace{1 \cdot a}_{a} + \underbrace{(-1)a}$$

Aha! $(-1)a$ is the additive inverse of a (which is unique)

Δ-ineq $\quad |a+b| \le |a| + |b|$

Pf: $\quad -|a| \le a \le |a| \quad$ [from (3) above.]

$+ \quad -|b| \le b \le |b|$

$$\underbrace{-|a| - |b|}_{-(|a|+|b|)} \le a + b \le |a| + |b|$$

So $\quad |a+b| \le |a| + |b|$ ✓ $\quad$ [from (3) again]

Conseq of Δ-ineq: $|a+b| \geq \big||a|-|b|\big|$

Particularly useful when $a+b$ is a denominator.

Replace b by $-b$. See $\quad |a-b| \geq \big||a| - \underbrace{|b|}_{=|-b|}\big|$

Why: $\quad a = a-b+b$

$$|a| \leq |a-b| + |b|$$

$$|a| - |b| \leq |a-b| \quad (*)_1$$

$\quad b = b-a+a$

$$|b| \leq \underbrace{|b-a|}_{=|a-b|} + |a|$$

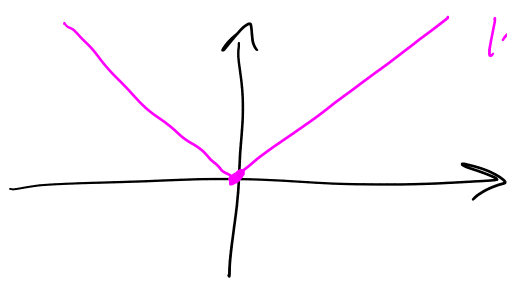
$$|b| - |a| \leq |a-b| \quad (*)_2$$

Case $|b| > |a|$ $\quad (*)_2$: $|a-b| \geq \underbrace{|b|-|a|}_{=\big||b|-|a|\big|}$ ✓

Case $|b| = |a|$ $\quad |a-b| \geq \underbrace{\big||a|-|b|\big|}_{0}$ Duh! ✓

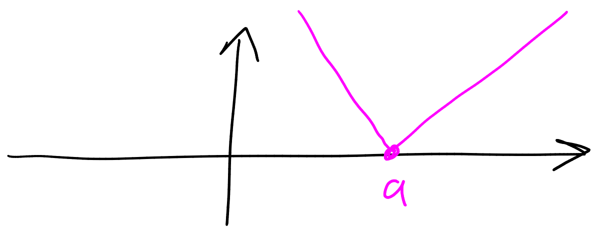
Case $|b| < |a|$ $\quad (*)_1$: $|a-b| \geq \underbrace{|a|-|b|}_{=\big||a|-|b|\big|}$ ✓

Graphs



$|x|$

Cont fcn.
Not diff'ble at $x=0$.



$|x-a|$

Δ-ineq: $|a+b| \le |a| + |b|$ <u>numerator</u> estimate

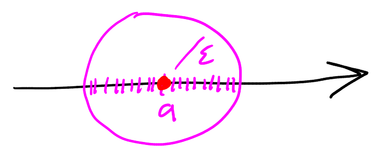
$|a+b| \ge \big||a| - |b|\big|$ <u>denomenator</u> est

$$\left|\frac{a+b}{c+d}\right| \le \frac{|a|+|b|}{\big||c|-|d|\big|}$$

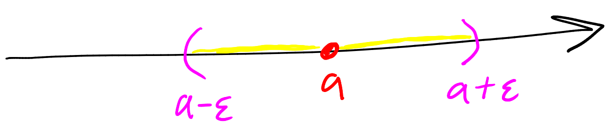
Useful when estimating $\left|\frac{f(x)}{g(x)}\right| \le \frac{\operatorname{Max}|f|}{\min|g|}$

Topology of $\mathbb{R}$: Start: ε-neighborhood of a:

$$V_\varepsilon(a) = \{x : |x-a| < \varepsilon\}$$



Remark $D_\varepsilon(a)$ in $\mathbb{C}$ in MA 425



<u>Thm</u>: If $x \in V_\varepsilon(a)$ for all $\varepsilon > 0$ (no matter how small), then $x = a$.

Book: Lemma: There is no smallest positive real number.

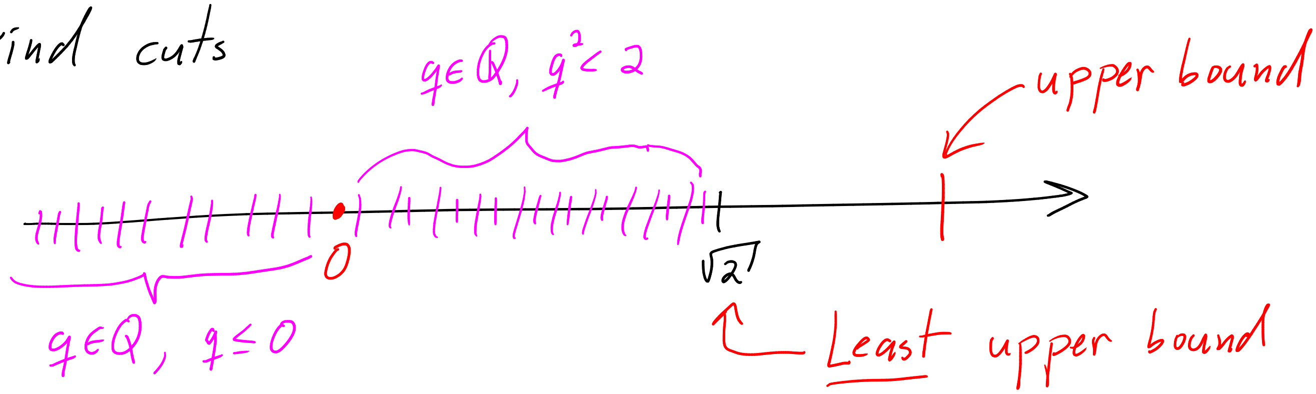
<u>Pf</u>: Suppose the hyp. Assume $x \ne a$. Then $|x-a| = c \ne 0$.

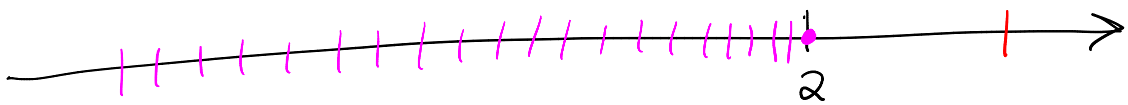
Oops. If $0 < \varepsilon < c$, then $x \notin V_\varepsilon(a)$. $\Downarrow$ So $x = a$.

Next time: Least upper bound principle (lub), greatest lower bound princ (glb).

Deep fact! Why calculus works.

Dedekind cuts





still called least upper bound. even though 2 is in set.

Upper bound $M \geq$ everything in Set