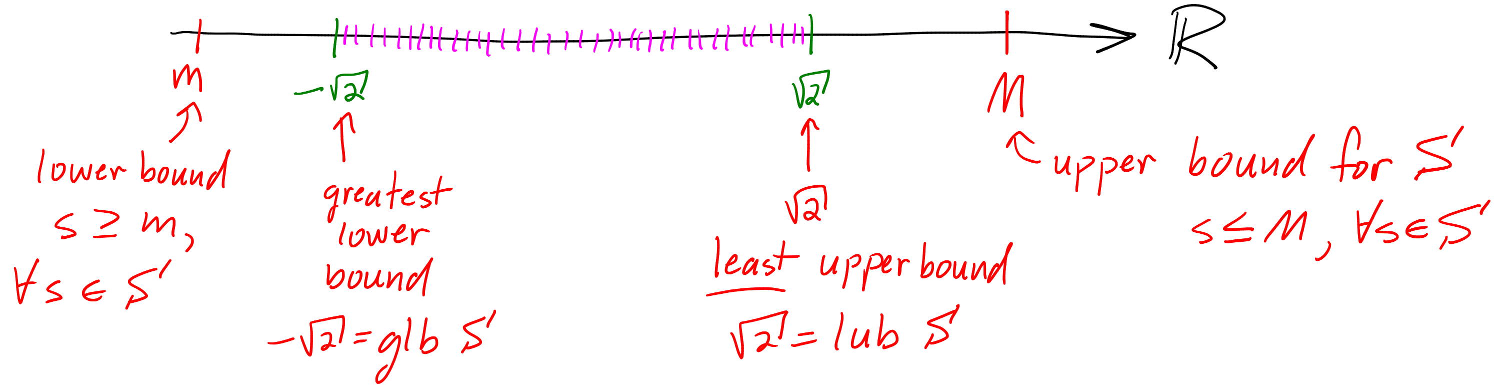
Lesson 6 Sup's and inf's (2.3)

$S' = \{p \in \mathbb{Q} : p^2 < 2\}$



Words 1) S' bounded from above means $\exists$ an upper bound M

2) S' " " below " " lower " m

3) S' called bounded when $\exists$ both upper & lower bnd.

Ex: $\mathbb{Q}$ unbounded. $\mathbb{Q}^+$ bdd below, not above.

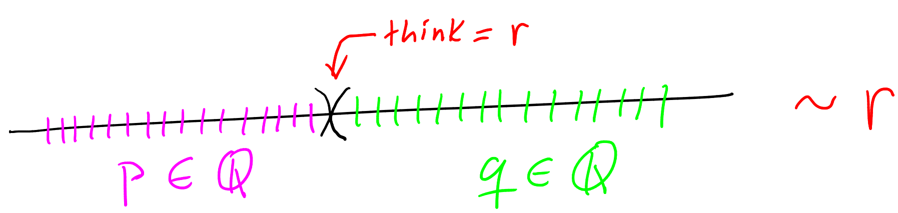
$\mathbb{Q}^-$ bdd above, not below.

Huge fact If S' bdd above, then there exists a least upper bound.

Sometimes called the "Least upper bound principle".
Book calls it the "completeness property of $\mathbb{R}$."

[Fact: It doesn't follow from the field axioms plus order axioms from $\mathbb{R}$. Need it as an axiom.

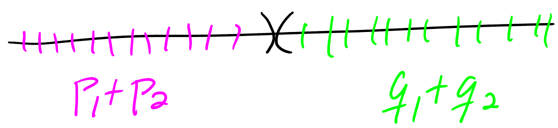
Dedekind



$p < q$

lub property lub $S' = \underbrace{\bigcup \{\text{left hand sides cuts } r \in S'\}}_{\text{left hand side of cut for lub } S'}$

Dedekind came up with a "model" for $\mathbb{R}$ where lub prop holds from def^{n}s!

EX: $r_1 + r_2 \sim$ 

Work!

Suppose M is an upper bnd for S'.

__Fact__: $M = \text{lub } S' \iff$ for every $\varepsilon > 0$, there is a $s \in S'$ such that $s > M - \varepsilon$.

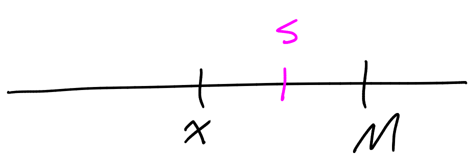
__Why__:

($\Rightarrow$) Suppose upper bnd M is the lub. Let $\varepsilon > 0$.

Hmmm. $M - \varepsilon$ is $< M$. M is the __least__ ub.

So $M - \varepsilon$ is __not__ an upper bnd. __Not__ true that $s \le M - \varepsilon$ for all s. Must be an s with $s > M - \varepsilon$. ✓

($\Leftarrow$) M is an upper bnd and ε statement holds for M.



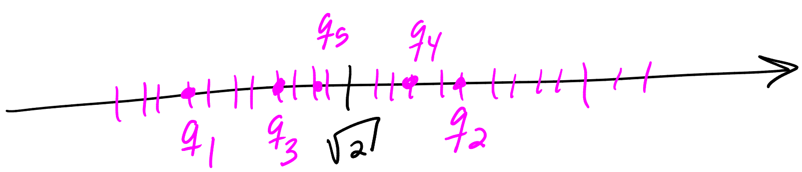
If $x < M$, let $\varepsilon = M - x > 0$. Get s with $\underbrace{x < s}_{\text{Aha! } x \text{ is not an upper bound for } S'} < M$

Hmmm. M is a ub. Anything less isn't. So $M = \text{lub } S'$.

EX: $S' = \{\pi\}$. Statements seem silly, but are still true.

Fancy words: $\text{lub} = \text{Supremum} = \text{Sup}$
$\text{glb} = \text{infimum} = \text{inf}$

Remark Freshman calculus: $\mathbb{R}$ "complete" means that every Cauchy sequence has a limit. The lub property is equivalent to that.

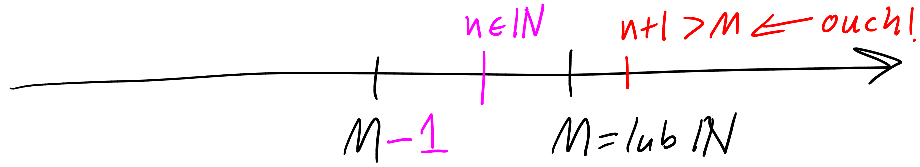


Cauchy seq means q_n's getting closer and closer together.
Completeness means q_n's must converge to a limit.

Archimedean property If $r \in \mathbb{R}$, there is a $n \in \mathbb{N}$ with $n \geq r$.

Fun fact This follows from lub prop.

Why: Let $r \in \mathbb{R}$. Suppose Arch Prop failed for r. Then there is no n with $n \geq r$. So $n < r$ for all n! So r is an upper bound for $\mathbb{N}$.
Let $M = \text{lub } \mathbb{N}$. Aha! Take $\varepsilon = 1$.



M <u>not</u> an upper bnd

↯. So no such r

Remark Related to: No smallest positive real number.