Lesson 7 Applications of the $\underbrace{\text{least upper bound}}_{\text{lub = Sup (Supremum)}}$ property (2.4)

lub prop equiv to $\underbrace{\text{glb}}_{\inf\,=\,\text{infinum}}$ (greatest lower bound) prop

Archimedean property of $\mathbb{R}$ Given a real number $r \in \mathbb{R}$, there is a positive integer $n \in \mathbb{N}$ with $n \geq r$.

Last time: Showed lub prop $\Rightarrow$ Arch. prop.

[Related to: No biggest pos integer.]

Cor of Arch prop $\inf \underbrace{\{\frac{1}{n} : n \in \mathbb{N}\}}_{S'} = 0$

Pf: Note that S' is bndd below by 0. So S' has a glb = inf. Let $r = \inf S'$.

Since 0 is a lower bnd, and r is the greatest l.b., we see that $r \geq 0$.

If $r > 0$, use Arch prop to get $n \in \mathbb{N}$ with

$0 < \frac{1}{r} < n$. [Take $n+1$ if $\frac{1}{r} = n$ in A. prop]

Then $\frac{1}{n} < r$. ← Oops! r is not even a lower bnd. ↯

Contradiction. So $r = 0$. ✓

Cor If $r \in \mathbb{R}^+$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.

Pf: Did this in pf of previous Cor.

<u>Cor</u> If $r \in \mathbb{R}^+$, there is an $n \in \mathbb{N}$ such that

$n-1 \le r \le n$, i.e., $r \in [n-1, n]$.

<u>Pf</u>: Let $S' : \{ m \in \mathbb{N} : m > r \}$. Arch prop $\Rightarrow$

S' <u>not</u> empty. Well-ordering property of $\mathbb{N}$

implies S' has a <u>smallest</u> element n_0.

Then $n_0 - 1$ is <u>not</u> in S'. So $n_0 - 1 \le r$.

In fact, see $r \in [n_0 - 1, n_0) \subset [n_0 - 1, n_0]$. ✓

Area of math: Non-Archimean fields. ($\mathbb{R}$ is an Arch. field)

<u>Consequence of Arch prop</u> Rational numbers are "dense" in $\mathbb{R}$,

meaning, given $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, there is

a $\frac{p}{q} \in \mathbb{Q}$ with $r_1 < \frac{p}{q} < r_2$. In fact, there

are <u>infinitely</u> many!

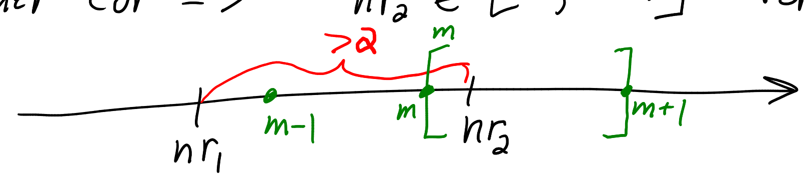
Read book's pf on p.44.

<u>Pf</u>: $r_1 < r_2$. So $r_2 - r_1 > 0$. Cor above $\Rightarrow$

$\exists\, n \in \mathbb{N}$ with $\frac{1}{n} < \frac{r_2 - r_1}{2}$.

So $2 < n(r_2 - r_1) = nr_2 - nr_1$

Hmmm. Other Cor $\Rightarrow$ $nr_2 \in [m, m+1]$ for some $m \in \mathbb{N}$.



Get $nr_1 < m-1 < nr_2$

$$r_1 < \underbrace{\frac{m-1}{n}}_{\in \mathbb{Q}} < r_2 \quad \checkmark$$

Cor Irrational numbers are dense too. Given $r_1 < r_2$ in $\mathbb{R}$, there is an irrational number $\tilde{r}$ with $r_1 < \tilde{r} < r_2$.

Pf: $r_1 < r_2$

So $\frac{r_1}{\sqrt{2}} < \frac{r_2}{\sqrt{2}}$. $\exists\ \frac{p}{q} \in \mathbb{Q}$ with

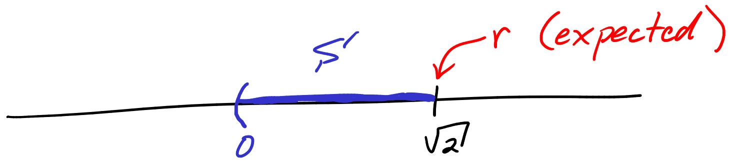
$$\frac{r_1}{\sqrt{2}} < \frac{p}{q} < \frac{r_2}{\sqrt{2}}$$

So $r_1 < \underbrace{\frac{p}{q}\sqrt{2}}_{\text{irrational!}} < r_2$ $\left[\ \frac{p}{q}\sqrt{2} = \frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q} \Rightarrow \sqrt{2} \in \mathbb{Q}.\ \lightning.\ \right]$

Square root of 2 exists! (via the lub property) p. 43.

Let $S' = \{ s \in \mathbb{R}^+ : s^2 < 2 \}$. S' is bounded above by 3.

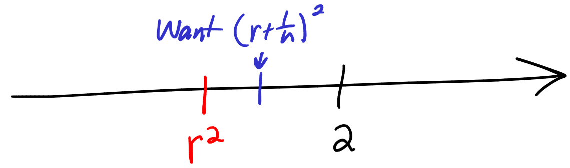
So it has a Sup = lub. Let $r = \operatorname{Sup} S'$

Think 

Claim: $r^2 = 2$. (Think Trichotomy prop)

Case $r^2 < 2$ Hmmm. $\left(r + \frac{1}{n}\right)^2 =$

$$r^2 + \underbrace{2r\left(\tfrac{1}{n}\right) + \left(\tfrac{1}{n}\right)^2}_{\text{make this } < 2-r^2}$$



Note that $\left(\frac{1}{n}\right)^2 < \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

$$2r\left(\tfrac{1}{n}\right) + \left(\tfrac{1}{n}\right)^2 < \underbrace{2r\cdot\tfrac{1}{n} + \tfrac{1}{n}}_{\frac{1}{n}(2r+1)} \overset{\text{want}}{<} (2-r^2)$$

Need $\quad \frac{1}{n}(2r+1) < (2-r^2)$

$$n > \frac{2r+1}{2-r^2} \quad \leftarrow \text{Use Arch. prop!}$$

$(r+\frac{1}{n})^2 < 2 \Rightarrow r+\frac{1}{n} \in S \Rightarrow r$ is not an upper bnd for S. $\lightning$

Get contradiction. So not case $r^2 < 2$.

Check $r^2 > 2$ case. Read p. 43. Similar.

Use $(r-\frac{1}{n})^2$.