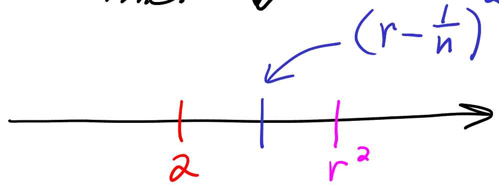
Lesson 8 Nested intervals and Cantor's big idea (2.5)

$\sqrt{2}$ exists: $S' = \{ s \in \mathbb{R}^+ : s^2 < 2 \}$ $r = \operatorname{Sup} S'$ (expect $= \sqrt{2}$)

$r^2 < 2$ done last time. $\lightning$

<u>Case $r^2 > 2$</u>



$$\left(r - \tfrac{1}{n}\right)^2 = r^2 - (2r)\tfrac{1}{n} + \underbrace{\tfrac{1}{n^2}}_{>0} > r^2 - \underbrace{(2r)\tfrac{1}{n}}_{\text{want } < (r^2 - 2)}$$

Need $(2r)\frac{1}{n} < r^2 - 2$

$$\frac{1}{n} < \frac{r^2 - 2}{2r} \leftarrow \text{positive}$$

$$n > \frac{2r}{r^2 - 2} \quad \checkmark \ \text{Arch. prop. Get } n.$$

Hmmm. If $s \in S'$, then $s^2 < 2 < \left(r - \frac{1}{n}\right)^2$

Ha! $s^2 < \left(r - \frac{1}{n}\right)^2 \leftarrow (\text{positive})^2$

So $s < \left(r - \frac{1}{n}\right)$ too.

What? $r - \frac{1}{n}$ is an <u>upper</u> <u>bound</u> for S'.

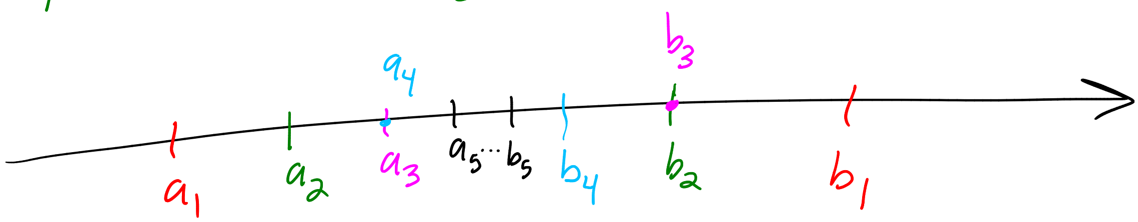
That's less than r and r is the <u>least</u> upper bound. $\lightning$ So can't be that $r^2 > 2$. So r^2 must $= 2$. $\checkmark$

Fact: $\sqrt{s}$ for $s \in \mathbb{R}^+$ exist too.

Nested closed intervals theorem. $I_n = [a_n, b_n]$

$= \{x : a_n \le x \le b_n\}$

$I_1 \supset I_2 \supset I_3 \supset \cdots$



Thm: $\bigcap_{n=1}^{\infty} I_n \ne \emptyset$. So $\exists\, r \in \bigcap_{n=1}^{\infty} I_n$,

meaning that there is an r such $r \in I_n$ for $\forall n$.

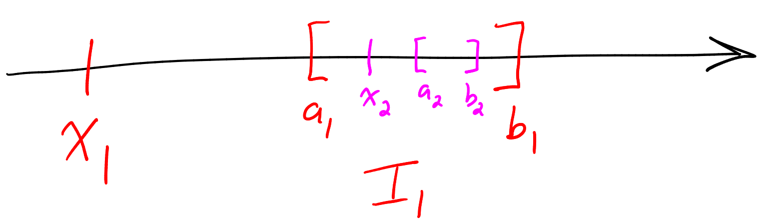
Sketch of pf: $\sup\{a_n\}_{n=1}^{\infty} \le r \le \inf\{b_n\}_{n=1}^{\infty}$

Pf next time.

Cantor knew this. Assume it for now.

Cantor's first pf that $\mathbb{R}$ is uncountable.

Suppose $\mathbb{R}$ is countable. It's infinite because $\mathbb{N} \subset \mathbb{R}$. So $\exists\, \varphi : \mathbb{N} \xrightarrow[\text{onto}]{1-1} \mathbb{R}$. $x_n = \varphi(n)$



Step 1 $x_1 \in \mathbb{R}$. Pick closed interval $[a_1, b_1]$

such that $x_1 \notin [a_1, b_1]$. $\leftarrow I_1$

Step 2 Pick interval $[a_2, b_2] \subset [a_1, b_1]$

such that $x_2 \notin [a_2, b_2]$. $\leftarrow I_2$

Remark: If x_2 is outside $[a_1, b_1]$, can

take $I_2 = I_1$.

Step 3: $[a_3, b_3] \subset [a_2, b_2]$ such that $x_3 \notin [a_3, b_3]$.

$\vdots$

Aha! Nested closed interval thm gives $r \in \bigcap_{n=1}^{\infty} I_n$.

So $r \in I_n$ for each n.

So $r \neq x_n$ because $x_n \notin [a_n, b_n]$ and r is $\in$.

r does not get hit by φ. φ is not onto.

Cantor's theorem: No map $S \xrightarrow[\text{onto}]{} P(S)$ $\leftarrow$ power set of S

$P(S)$ is a bigger infinity than S.

$P(P(S))$ is even bigger.

$\vdots$

There is no biggest infinity!

<u>Russell's paradox</u> Let U = Set of all sets.

Let $S' = \{A \in U : A \notin A\}$.

Is $S' \in S'$?

Either $S' \in S'$ or $S' \notin S'$.

If $S' \in S'$. Doesn't satisfy $A \notin A$.

So $S' \notin S'$! $\lightning$

If $S' \notin S'$. $A \notin A$. So $S' \in S'$. $\lightning$.

Horror! Set theory might be inconsistent.

Zermelo-Frankl axioms of set theory.

Must be careful about "sets" that are <u>too</u> big.

Axiom of choice.