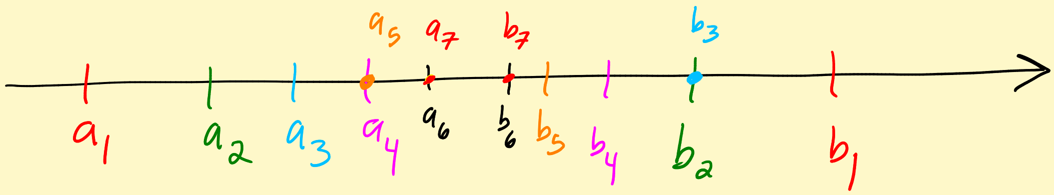
Lesson 9 More about nested intervals (2.5)

$$\underset{I_1}{[a_1, b_1]} \supset \underset{I_2}{[a_2, b_2]} \supset \underset{I_3}{[a_3, b_3]} \supset \cdots$$



$$a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1$$

Pf of N. (Closed) I. T. $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

$[a_n, b_n] \subseteq [a_1, b_1]$. So $a_n \le b_n \le b_1$.

Hence b_1 is an upper bound for $\{a_n\}_{n=1}^{\infty}$.

So $\{a_n\}_{n=1}^{\infty}$ has a least u.b. $= r = \sup\{a_n\}_{n=1}^{\infty}$.

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots$$

Know: $a_1 \le a_2 \le a_3 \le \cdots$

$$[a_{n+k}, b_{n+k}] \subset [a_n, b_n]$$

So $a_n \le a_{n+k} \le b_{n+k} \le b_n$

$a_{n+k} \le b_n$ for $k = 0, 1, 2, 3, \ldots$

But $a_1 \le a_2 \le \cdots \le a_n$. So all a's $\le b_n$.

So b_n is an u.b. for $\{a_n\}_1^\infty$ too.

r is the least u.b. so $r \le b_n$ for all n.

So b_n's are bnd below. $\{b_n\}_{n=1}^\infty$ have a greatest l.b. and $r \le \inf\{b_n\}_1^\infty$.

Aha! $a_n \le r \le b_n$. $r \in I_n$ $\forall n$.

So $r \in \bigcap_1^\infty I_n$. So $\ne \emptyset$.

Remark: $\sup\{a_n\} = r \le \inf\{b_n\}$.

If $\sup\{a_n\} = r_1 < r_2 = \inf\{b_n\}$,

then $[r_1, r_2] \subset \bigcap_1^\infty I_n$.

Thm: In nested closed interval thm, if

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$$

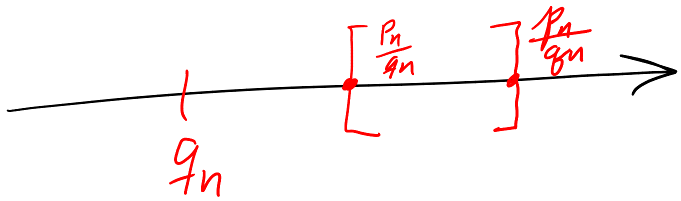
(meaning that $\lim_{n\to\infty} b_n - a_n = 0$, next section), then

$\bigcap_1^\infty I_n = \{r\}$, a single pt.

EX $\mathcal{S}$ = set of sets that contain at least one element.
($\mathcal{S}$ is not empty. $\{\emptyset\} \in \mathcal{S}$)

$\delta' \in S'$.

Why can't we use Cantor's idea to show $\mathbb{Q}$ is uncountable?



$\bigcap_{1}^{\infty} I_n$ contains r, but r might be irrational, like $\sqrt{2}$!

<u>EX:</u> 1) $I_n = (0, \frac{1}{n})$. $\bigcap_{1}^{\infty} I_n = \emptyset$

2) $I_n = (0, \frac{1}{n}]$. $= \emptyset$ too.

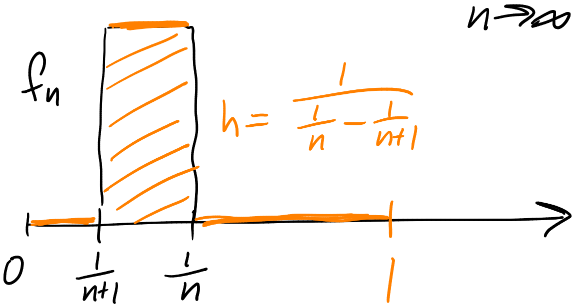
So <u>closed</u> is critical for the N.I.T.

Why are we being <u>so</u> picky and careful.

<u>Ques:</u> Suppose $f_n(x) \geq 0$ and $\lim_{n\to\infty} f_n(x) = 0$

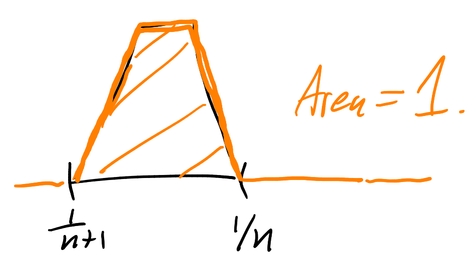
for each x. Is $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$?

No!



$\int_0^1 f_n(x)\,dx = 1$

Could cook up continuous ones too



Cantor's second pf that $\mathbb{R}$ is uncountable.

$$\varphi : \mathbb{N} \xrightarrow{\text{onto}} [0,1] \qquad \varphi(n) = x_n$$

$$\begin{aligned} x_1 &= .d_{11}\, d_{12}\, d_{13}\, d_{14} \cdots & D_1 &\neq d_{11} \\ x_2 &= .d_{21}\, d_{22}\, d_{23}\, d_{24} \cdots & D_2 &\neq d_{22} \\ x_3 &= .d_{31}\, d_{32}\, d_{33}\, d_{34} \cdots & D_3 &\neq d_{33} \\ &\vdots & &\vdots \end{aligned}$$

$r = .D_1 D_2 D_3 \cdots$ does not get hit by φ.

Technical point (see 2.5). Are decimal expansions unique?

3	4
3.1	3.2
3.14	3.15
3.141	3.142
3.1415	3.1416
⋮	⋮

$$\sup = \pi = \inf$$

Technical pt. $\quad .999999\ldots = 1.0$

$$S_n = 1 + r + r^2 + \cdots + r^n$$
$$r S_n = \quad r + r^2 + \cdots + r^n + r^{n+1}$$

$$(1-r)S'_n = 1 - r^{n+1}$$

$$S'_n = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$$

$$.9999\cdots 9_n$$

$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n}$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{n-1}}\right)$$

$$= \frac{9}{10}\left[\frac{1}{1-1/10} - \frac{1/10^n}{1-1/10}\right]$$

$$= 1 - \frac{1}{10^n}$$

$.9, .99, .999, \ldots$ is an increasing seq.

See its Sup $= 1$.

$$.99\ldots = 1$$

Toss out repeating 9's in decimal exp

EX $.1234999999 \leftarrow .12350000$ instead.

Then decimal exp is unique.

[Note: Decimal expansion is repeating $\Leftrightarrow$ number is <u>rational</u>]