Lesson 10 Sequences, limits (3.1)

HWK 3 due tonight
HWK 4 posted, due next Wed.

EX $x_n = \frac{1}{n}$

Archimedes princ $\Rightarrow$ "$x_n \to 0$ as $n \to \infty$"

Seq : $\varphi : \mathbb{N} \to \mathbb{R}$ $\quad x_n = \varphi(n)$.

Notation X, (x_n), $\underbrace{(x_n)_{n=1}^{\infty}}_{\text{my fav}}$, $\{x_n : n \in \mathbb{N}\}$.

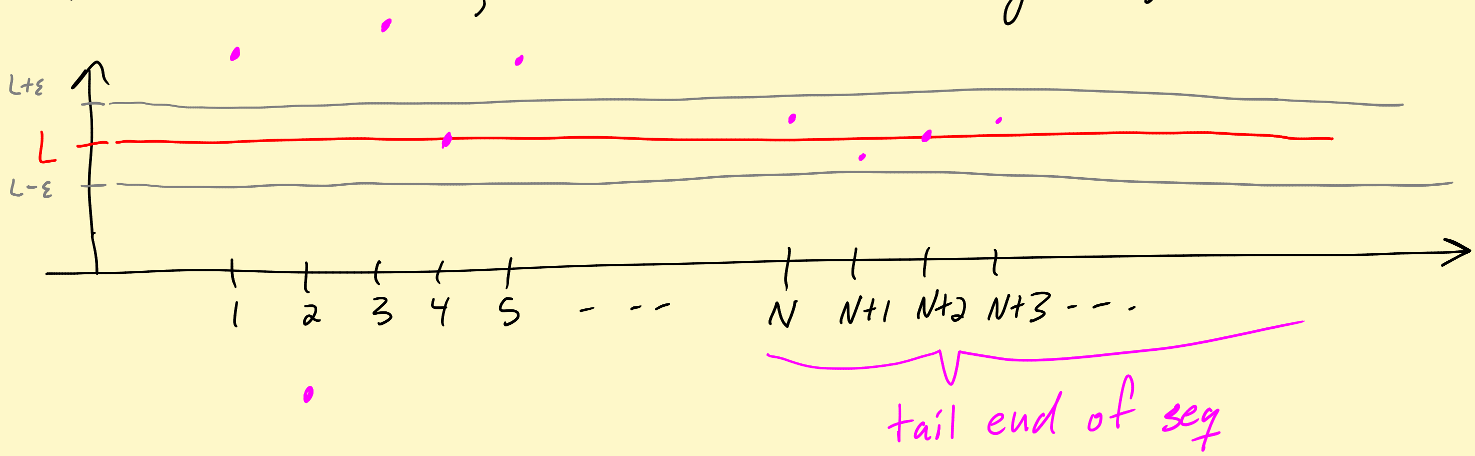
$x_n \to L$ as $n \to \infty$ same as $\lim_{n\to\infty} x_n = L$

Def^n $L \in \mathbb{R}$. $\lim_{n\to\infty} x_n = L$ means, given $\varepsilon > 0$, no matter how small, there is a pos int N such that

$|x_n - L| < \varepsilon$ when $n \geq N$.

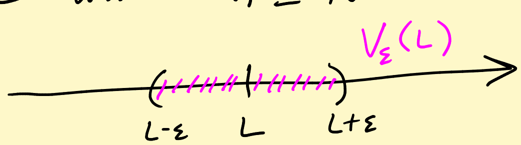
$> N$ gives same def.

Think: ε = error, or tolerance (for engineers).



Other ways to say $|x_n - L| < \varepsilon$ when $n \geq N$

$V_\varepsilon(L) = \{x : |x - L| < \varepsilon\}$



$x_n \in V_\varepsilon(L)$ when $n \geq N$.

or $L-\varepsilon < x_n < L+\varepsilon$ when $n \geq N$.

EX $x_n = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

because, given $\varepsilon > 0$,

$$|x_n - L| < \varepsilon$$
$$\left|\tfrac{1}{n} - 0\right| < \varepsilon$$
$$\frac{1}{n} < \varepsilon$$
$$n > \frac{1}{\varepsilon}$$

Archimedes princ! $\exists$ such an n.

Let N be first N such that $N > \frac{1}{\varepsilon}$.

Then, if $n \geq N$, then $n > N > \frac{1}{\varepsilon}$. Follow string of ineq backwards, get $\left|\frac{1}{n} - 0\right| < \varepsilon$ when $n \geq N$.

EX: Same string shows $\frac{(-1)^n}{n} \longrightarrow 0$ as $n \to \infty$.

$$\frac{(-1)^n \sin n}{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\left(\text{because } |(-1)^n \sin n| = |\sin n| \leq 1\right)$$

EX: $\frac{5n+2}{2n+7} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \frac{5 + \frac{2}{n}}{2 + \frac{7}{n}} \xrightarrow{\text{Guess}} \frac{5}{2}$ as $n \to \infty$.

Let $\varepsilon > 0$. $\left|\frac{5n+2}{2n+7} - \frac{5}{2}\right| < \varepsilon$ want

$$\left| \frac{2(5n+2) - 5(2n+7)}{2(2n+7)} \right| \overset{\text{want}}{<} \varepsilon$$

$$\left| \frac{-31}{2(2n+7)} \right| < \varepsilon$$

$$\frac{31/2}{2n+7} < \varepsilon$$

$$2n+7 > \frac{31}{2\varepsilon}$$

Need $n > \frac{31}{4\varepsilon} - \frac{7}{2}$.

Let $N =$ first one bigger than that.

<u>Ex</u> $x_n = \sqrt{n+1} - \sqrt{n} \longrightarrow 0$ as $n \to \infty$.

"$\infty - \infty$" $\leftarrow$ Fresh calc; Indeterminate form.

$$(\sqrt{n+1} - \sqrt{n}) \cdot \left[\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right] = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} \leftarrow \underbrace{(n+1) - (n)}_{1}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \overset{\text{want}}{<} \varepsilon$$

$$\sqrt{n+1} + \sqrt{n} > \sqrt{n}$$

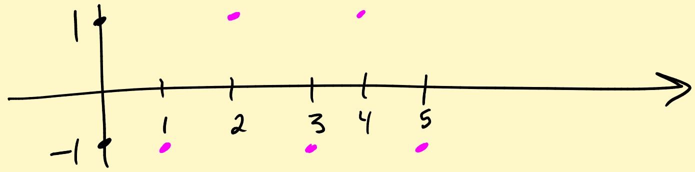
$$\frac{1}{\sim\sim\sim} < \frac{1}{\sqrt{n}}$$

$$\underbrace{\left(\frac{1}{\sqrt{n}}\right)^2}_{\frac{1}{n}} < \varepsilon^2$$

Need $n > 1/\varepsilon^2$. $N =$ first such n.

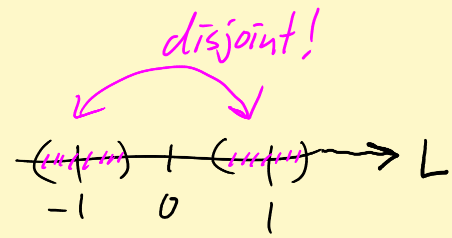
To show $x_n \not\to L$, have to come up with <u>just</u> <u>one</u> ε where this "game" fails.

<u>EX</u>: $x_n = (-1)^n$



Hmmm. Take $\varepsilon = \frac{1}{2}$ $\quad \left| (-1)^n - L \right| < \frac{1}{2}$

$$\begin{cases} n \text{ even} & |L-1| < \frac{1}{2} \\ n \text{ odd} & |L+1| < \frac{1}{2} \end{cases}$$



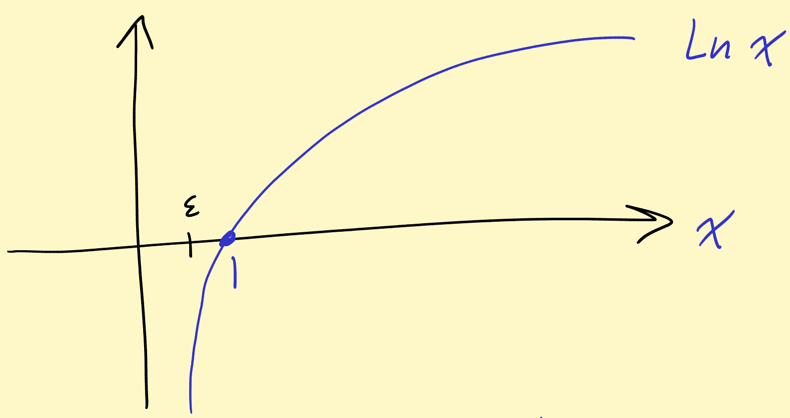
No such L.

$\varepsilon = 1$ works too.

<u>EX</u> $0 < r < 1$. Then $r^n \to 0$ as $n \to \infty$.

Want $r^n < \varepsilon$.

$0 < r^n < 1$



If $\varepsilon \geq 1$, all $r^n < \varepsilon$.

If $0 < \varepsilon < 1$.

Ln x is <u>strictly</u> increasing

$$r^n < \varepsilon \iff \underbrace{\ln r^n}_{n \underbrace{\ln r}_{<0}} < \underbrace{\ln \varepsilon}_{<0} \quad \text{because}$$

$$\underbrace{-n \ln r} > -\ln \varepsilon$$

$$n\underbrace{(-\ln r)}_{>0} > \underbrace{(-\ln \varepsilon)}_{>0}$$

Need $n > \dfrac{-\ln \varepsilon}{-\ln r} = \dfrac{\ln \varepsilon}{\ln r}$

Take N = first one. ✓

Other famous limits

1) $c > 0$. $\lim_{n\to\infty} c^{1/n} = 1$

2) $\lim_{n\to\infty} n^{1/n} = 1$

Cheating to use log's and exponential today.

Book cleverly uses Bernoulli's ineq: If $x > -1$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

(Math induction. Page 30.) to prove above facts.

Cheat $n^{1/n} = e^{\ln n^{1/n}} = e^{\frac{1}{n}\ln n} \longrightarrow e^0 = 1$ because

$$\frac{1}{n}\ln n = \frac{\ln n}{n} \xrightarrow{\text{L'H}} \lim_{n\to\infty} \frac{\frac{d}{dn}(\ln n)}{\frac{d}{dn}(n)} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and e^x is continuous at $x=0$.