Lesson 11 Basic limit facts (3.2)

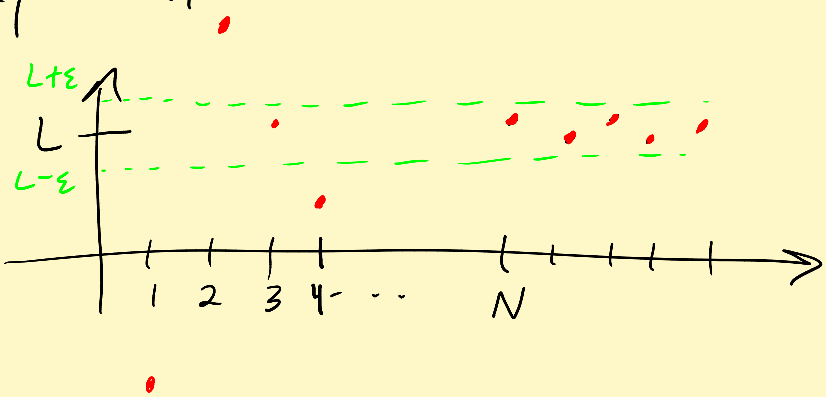
Ex: $\sin n$ $\left[\frac{p}{q} \approx \pi \quad p \approx q\pi \leftarrow \sin q\pi = 0 \right]$

Weird series! But at least it's bounded because $|\sin\theta| \leq 1$ for all θ.

Def^n $(x_n)_{n=1}^{\infty}$ is a bounded seq means, there is an $M>0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Thm A convergent seq must be bounded

Why $x_n \to L \in \mathbb{R}$



Take $\varepsilon = 1$. Get N such that $|x_n - L| < \underbrace{1}_{\varepsilon}$ when $n \geq N$.

Today's trick: add and subtract something clever and use Δ-ineq.

$$|x_n| = |x_n - L + L| \leq \underbrace{|x_n - L|}_{<1,\ n \geq N} + |L|$$

$\leq 1 + |L|$ when $n \geq N$.

$$M = \operatorname{Max}\left(|x_1|, |x_2|, \ldots, |x_{n-1}|, 1+|L|\right)$$

Read 3.2 carefully.

Thm $\ell_n \to \ell$ as $n \to \infty$ for $\ell = x, y$

1) $\underbrace{\lim_{n\to\infty} (x_n + y_n)}_{\text{limit exists}} = x + y$

2) $\lim_{n\to\infty} x_n y_n = xy$

3) $\lim_{n\to\infty} c x_n = c x$ when c is a const.

4) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$ provided that $y \neq 0$.

Pf 1)

$$|(x_n + y_n) - (x + y)| =$$

$$|(x_n - x) + (y_n - y)|$$

$$\leq \underbrace{|x_n - x|}_{<\frac{\varepsilon}{2}} + \underbrace{|y_n - y|}_{\varepsilon/2}$$

$\frac{\varepsilon}{2}$ Trick

when $n \geq N_x$ when $n \geq N_y$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ when $n \geq \mathrm{Max}(N_x, N_y)$.

2) $|x_n y_n - xy| =$ Add and subtract something clever.

$$|x_n y_n - x y_n + x y_n - xy|$$

$$\leq |x_n y_n - x y_n| + |x y_n - xy|$$

$$\leq \underbrace{|y_n|}_{\substack{\leq M \\ \text{since } y_n \to y}} \underbrace{|x_n - x|}_{\substack{< \frac{\varepsilon}{2M} \\ n \geq N_x}} + |x| \underbrace{|y_n - y|}_{\substack{\frac{\varepsilon}{2|x|} \\ n \geq N_y}} < \varepsilon$$

when $n \geq \mathrm{Max}(N_x, N_y)$. Oops! Works when $x \neq 0$.

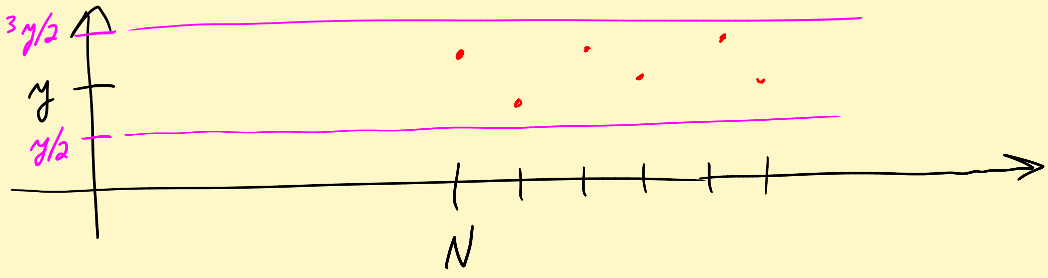
Aha! If $x = 0$, that term $= 0$!

3) $\dfrac{x_n}{y_n} = x_n \cdot \left(\dfrac{1}{y_n}\right)$

Reduce quotient case to product case using

<u>Lemma:</u> $\frac{1}{y_n} \to \frac{1}{y}$ provided that $y \neq 0$.

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| = \frac{|y - y_n|}{|y_n||y|}$$



$$\big||y| - |y_n|\big| \le |y - y_n| < \frac{|y|}{2}, \quad n \ge N_1$$

$$\big||y| - |y_n|\big| < \frac{|y|}{2}$$

So $$-\frac{|y|}{2} < |y| - |y_n| < \frac{|y|}{2}$$

$$-\frac{3|y|}{2} < -|y_n| < -\frac{|y|}{2}$$

$$\frac{3|y|}{2} > \boxed{|y_n| > \frac{|y|}{2}}$$

Back to $$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y - y_n|}{|y_n||y|} < \frac{|y - y_n|}{\left(\frac{|y|}{2}\right)(|y|)}$$

$$|y_n| \ge \frac{|y|}{2} \text{ if } n \ge N_1$$

Done if $|y - y_n| < \varepsilon\left(\frac{|y|^2}{2}\right)$ when $n \ge N_2$:

Take $n \ge \operatorname{Max}(N_1, N_2)$. ✓

EX $$\frac{5n - 7}{2n^2 + 3} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$ n^2 is biggest power

$$= \frac{\frac{5}{n} - \frac{7}{n^2}}{2 + \frac{3}{n^2}}$$

$$\longrightarrow \frac{5\cdot 0 - 7\cdot 0^2}{2 + 3\cdot 0^2} = \frac{0}{2} = 0$$

$\left(\frac{1}{n}\right) \to 0$

$\frac{5}{n} \to 5 \lim \frac{1}{n} = 0$

$\frac{1}{n^2} = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \to 0\cdot 0 = 0$

Easy things 1) $x_n \to 0$ as $n \to \infty$ and y_n bounded, then $x_n y_n \to 0$ as $n \to \infty$.

$$|x_n y_n - 0| = \underbrace{|x_n|}_{\substack{< \frac{\varepsilon}{M} \\ n \geq N}} \underbrace{|y_n|}_{\leq M} < \varepsilon, \quad n \geq N$$

Ex: $\frac{(-1)^n \sin n}{n} \to 0$

2) $x_n \to x$, then $|x_n| \to |x|$ as $n \to \infty$.

Why: $\big||x_n| - |x|\big| \leq |x_n - x| < \varepsilon \quad n \geq N$

3) $x_n \geq 0$ and $x_n \to x$ as $n \to \infty$.

$\boxed{\text{Then } x \geq 0.}$ And

$\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

Why $\left(\sqrt{x_n} - \sqrt{x}\right) \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$

$$\left|\sqrt{x_n}-\sqrt{x}\right| = \frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$$

Aha! Make $|x_n - x| < \sqrt{x}\,\varepsilon$ if $n \geq N$.

Oops! What if $x=0$? Turns out to an easy case.
Think about it.

Read about the "Squeeze theorem."