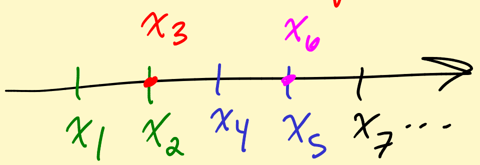
Lesson 12 The monotone convergence theorem

Office hour on Tuesday cancelled this week. Questions on Piazza.

"An increasing sequence that is bounded from above converges (to the supremum)."

$x_1 \le x_2 \le x_3 \le \cdots$ increasing

$x_1 \ge x_2 \ge x_3 \ge \cdots$ decreasing



Some books increasing = "non-decreasing", decreasing = "non-increasing"

"Strictly increasing" = $<$ in place of $\le$. etc.

Thm 1) $(x_n)_{n=1}^{\infty}$ is an increasing seq that is bounded from above, then $\lim_{n\to\infty} x_n$ exists and is equal to $\sup\{x_n : n \in \mathbb{N}\}$.

2) (x_n) decreasing, bounded from below, then $\lim_{n\to\infty} x_n = \inf\{x_n\}_{n=1}^{\infty}$.

Def^n: Monotone means increasing or decreasing.

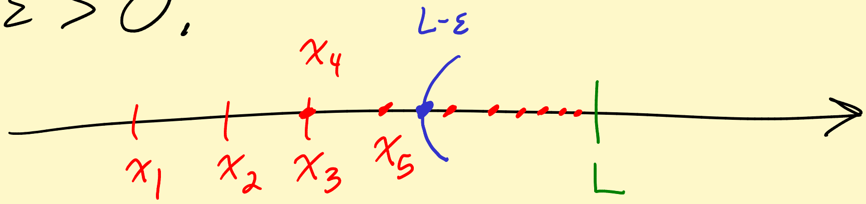
Summary: A monotone seq is convergent $\iff$ it is bounded.

Pf of (1): Suppose $(x_n)_1^{\infty}$ is increasing, bounded above.

It has an upper bound; so it a has least upper bound

$(= \sup\{x_n\}_1^\infty)$. Let $L = \sup\{x_n\}_1^\infty$.

Let $\varepsilon > 0$.



Claim: There is an x_N in $(L-\varepsilon, L]$.

If not, $x_n \le L-\varepsilon$ for all n. Oops! $L-\varepsilon$ would be an <u>upper</u> bound for $\{x_n\}_1^\infty$.

But $L-\varepsilon < L$ and L was the <u>least</u> upper bound. ↯

So $\exists\ x_N \in (L-\varepsilon, L]$. Aha! Since

$$x_N \le x_{N+1} \le x_{N+2} \le \cdots \le L$$

we see that $|x_n - L| = L - x_n < \varepsilon$ for $n \ge N$. ✓

Remark: An increasing seq is bndd below by x_1, a decreasing seq bndd above by x_1.

<u>EX</u>: $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^\infty$

$$1 < 2 < 3 < \cdots$$

$$\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \cdots$$

$$\sqrt{\frac{1}{1}} > \sqrt{\frac{1}{2}} > \sqrt{\frac{1}{3}} > \cdots$$

Strictly decreasing seq bndd below by 0.

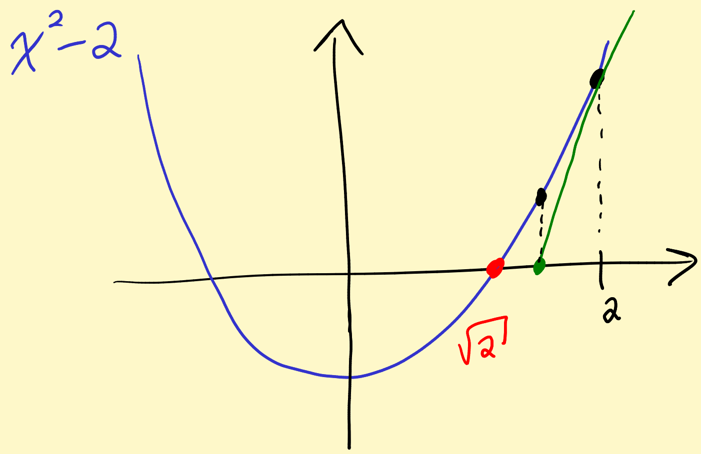
So limit exists and is equal to $0 = \inf \{\frac{1}{\sqrt{n}}\}_1^\infty$.

[Given $\varepsilon > 0$. $\exists\, N$ such that $N > \frac{1}{\varepsilon^2}$. ← Archimedian prop.

$$\frac{1}{N} < \varepsilon^2$$

$$\frac{1}{\sqrt{N}} < \varepsilon \quad]$$

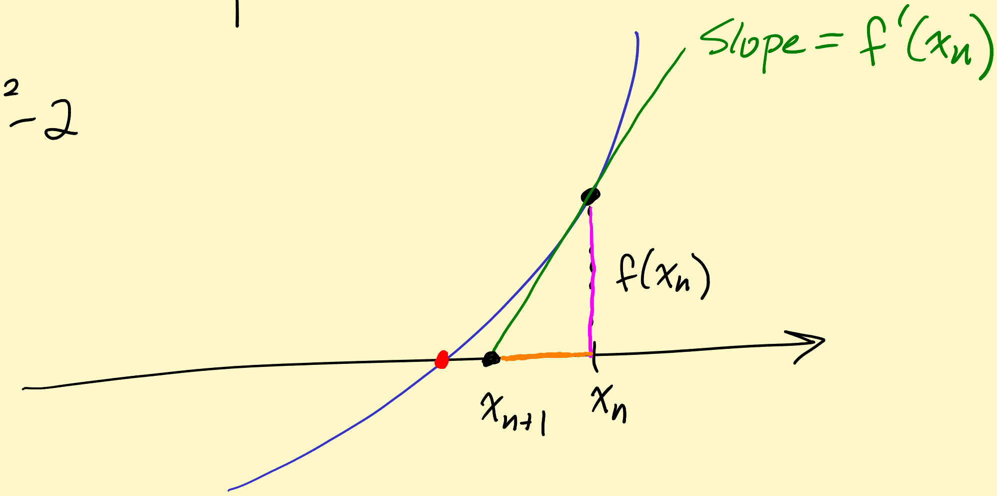
Newton's method for computing $\sqrt{2}$.



Aha! Decreasing seq, bounded below by 1. It has a limit!

Claim: limit $= \sqrt{2}$.

$f(x) = x^2 - 2$



Aha! $\quad f'(x_n) = \dfrac{f(x_n)}{x_n - x_{n+1}}$

$$\boxed{x_n - x_{n-1} = \frac{f(x_n)}{f'(x_n)} = \frac{x_n^2 - 2}{2x_n}} > 0$$

The x_n's are decreasing. (Play around with inequalities and induction to see $x_n^2 > 2$ for all n, so $x_n > 1$ for all n too.)

Newton's method: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - 2)}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}}$$

Cool thing: We know $x_n \to L$ as $n \to \infty$.

Also $x_{n+1} \to L$ as $n \to \infty$. Duh!

Take limits in red box: $$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$$

$$L = \frac{1}{2}L + \frac{1}{L}$$

know $L \geq 1$. $\neq 0$

$$\frac{1}{2}L = \frac{1}{L}$$

$$L^2 = 2$$ Bingo!

Book $\left(1+\frac{1}{n}\right)^n \longrightarrow e$ as $n \to \infty$.

$\left(1+\frac{1}{n}\right)^n$ is an increasing seq bounded above by 3.

Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \frac{1}{n^k}$$

Book: Cleverly splits up $\left(1+\frac{1}{n}\right)^n$, $\left(1+\frac{1}{n+1}\right)^{(n+1)}$

and shows $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < 3$

induction

Freshman calc: $\left(1+\frac{1}{n}\right)^n = e^{\ln\left(1+\frac{1}{n}\right)^n}$

$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \qquad \frac{0}{0}$$

$$\xrightarrow[\text{L'H}]{} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \left(\frac{-1}{n^2}\right)}{\left(\frac{-1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

Aha! $\ln\left(1+\frac{1}{n}\right)^n \longrightarrow 1$ and e^x is <u>continuous</u> at $x=1$.

So $\left(1+\frac{1}{n}\right)^n = e^{\ln\left(1+\frac{1}{n}\right)^n} \longrightarrow e^1 = e.$ ✓