Lesson 13 The Bolzano-Weierstraß Theorem (for sequences) (3.4)

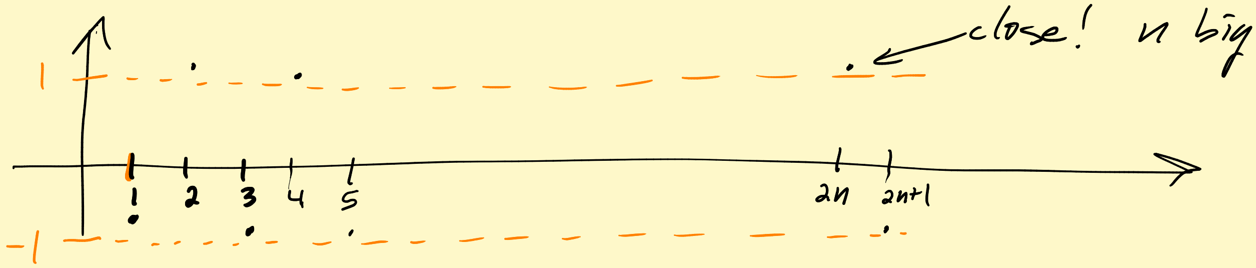
ss

HWK 5 posted. Due next Wed

Subsequences: EX: $(-1)^n + \frac{1}{n}$

n even: $(-1)^{2n} + \frac{1}{2n} = 1 + \frac{1}{2n}$

n odd: $(-1)^{2n+1} + \frac{1}{2n+1} = -1 + \frac{1}{2n+1}$



$\left(1 + \frac{1}{2n}\right)_{n=1}^{\infty}$, $\left(-1 + \frac{1}{2n+1}\right)_{n=1}^{\infty}$ are subsequences.

Def^n $(x_n)_{n=1}^{\infty}$ a seq. $n_1 < n_2 < n_3 < \cdots$

$(x_{n_k})_{k=1}^{\infty}$ is a subsequence of (x_n).

Easy fact If (x_n) converges to x, then every subseq of (x_n) converges to x.

EX: $x_n = \frac{1}{n}$. $\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right)$

$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots\right)$ (evens)

$\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots\right)$ (odds)

$\left(\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \cdots\right)$ (factorials)

$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \right) \qquad \left(\frac{1}{2^n}\right)$

They all conv to <u>zero</u> by easy fact.

<u>Fun application</u>. We showed that $b^n \to 0$ as $n \to \infty$ when $0 \le b < 1$ using Bernoulli's ineq.

$$1 > b > b^2 > b^3 > \cdots \ge 0$$

Decreasing seq bounded below.

So limit exists and $= \inf \{b^n\}_{n=1}^{\infty} \ge 0$

Sneeky trick: $1 > b^2 > b^4 > b^6 > \cdots$

is a Subsequence, so it converges to the <u>same</u> limit.

Write $L = \lim\limits_{n\to\infty} b^n$.

So
$$L = \lim_{n\to\infty} b^{2n} = \lim_{n\to\infty} (b^n)^2$$
$$= \left(\lim_{n\to\infty} b^n\right)^2 = L^2$$

$$L^2 - L = 0$$

$$L(L-1) = 0 \qquad \text{Aha!} \quad L = 0, 1.$$

Why can't it be 1? $\quad 1 > \underline{\underline{b}} > b^2 > \cdots$

$1 > \underline{\underline{b}}$ is an upper bound for all b^n, $n=1,2,\cdots$

So $\lim_{n\to\infty} b^n \le b < 1$.

Conclude $L=0$. ✓

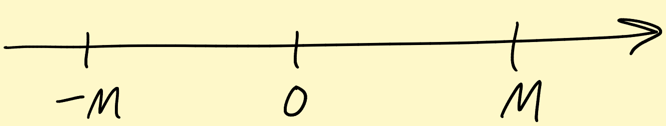
Similar argument yields $c^{1/n} \to 1$ as $n\to\infty$ for any $c>0$. [Cases: $c>1$, $0<c<1$, $c=1$ ✓]

<u>Bolzano-Weierstraß Thm</u> A <u>bounded</u> seq has a convergent subseq.

<u>Pf #2</u> Suppose (x_n) is bndd seq. Then

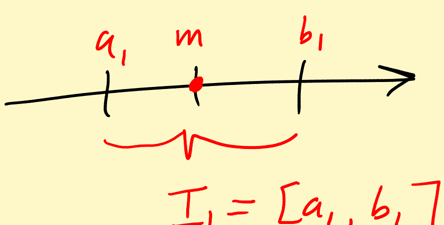
$$|x_n| \le M \text{ for all } n.$$

<u>Step 1</u>



<u>At least</u> one side $[-M,0]$ or $[0,M]$ contains infinitely many x_n's. Pick one. Call it I_1.

Pick $x_{n_1} \in I_1$.



$m = \dfrac{a_1+b_1}{2}$

Cut I_1 into two halfs: $[a_1, m]$, $[m, b_1]$.

Pick one with infinitely many x_n's in it.

Call it I_2. Since there are infinitely many $x_n \in I_2$, there must be one with a bigger index $n_2 > n_1$. Keep doing this. Get sub seq

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots$$

$$x_{n_k} \in I_k. \quad I_1 \supset I_2 \supset I_3 \supset \cdots$$

Nested closed interval thm: $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

Note: Cor to Nested interval thm:

If $|b_n - a_n| \to 0$ as $n \to \infty$, then

$$\bigcap_{k=1}^{\infty} I_k = \{p\}.$$

$$b_1 - a_1 = M$$

$$b_2 - a_2 = M/2$$

$$b_3 - a_3 = M/2^2$$

$$\vdots$$

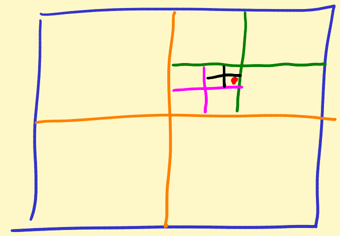
$$b_n - a_n = M/2^{n-1} \longrightarrow 0 \text{ as } n \to \infty.$$

$$p = \lim_{k \to \infty} x_{n_k}. \checkmark$$

make

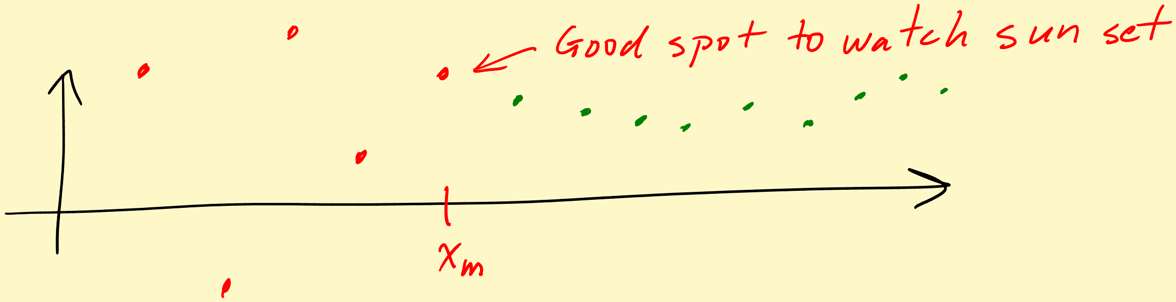
$$|p - x_{n_k}| \leq \frac{M}{2^{k-1}} < \varepsilon$$

Catch a tiger in Colorado.



Fun fact Every sequence has a monotone subseq.

Pf: (x_n) a seq. Call x_m a "vista point" if $x_m \geq x_n$ for $n \geq m$.



Case: (x_n) has infinitely many vista pts.

Let x_{n_1} be the first one.

Since there are infinitely many v. pts, there is one x_{n_2} with $n_2 > n_1$.

Since x_{n_1} is a vista pt. $x_{n_1} \geq x_{n_2}$.

Repeat! Get $x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq x_{n_4} \geq \cdots$

monotone! (decreasing).

Case 2: Only finitely many vista pts.

x_N is the last vista pt.

Let $n_1 = N+1$. x_{n_1} is not a vista pt.

Not case that $x_{n_1} \geq x_m$ for all $m \geq n_1$.

So there is $n_2 > n_1$ with $x_{n_2} > x_{n_1}$.

Repeat! Get $x_{n_1} < x_{n_2} < x_{n_3} < x_{n_4} < \cdots$

monotone! (strictly increasing).

Pf #1 of B-W Thm: (x_n) bndd seq.

Fun fact. There is a monotone subseq (x_{n_k}).

$$\underset{\text{bnd below}}{-M} \leq x_{n_k} \leq \underset{\text{bnd above}}{M}$$

Monotone conv thm says subseq conv.

HWK 4: 3.3: 12, 3.4: 7 12d is interesting!

$$\underbrace{\left(1+\tfrac{1}{n}\right)^n}_{2<\left(1+\frac{1}{n}\right)^n<3} \longrightarrow e \text{ as } n \to \infty$$

Def^n of e in book.

$2<\left(1+\frac{1}{n}\right)^n<3$ and $\left(1+\frac{1}{n}\right)^n$ is increasing. So limit exists.