Lesson 14 Cauchy sequences (the Cauchy criterion) (3.5)

End of 3.4: Lim Sup and Lim inf

(x_n) bounded seq. $\quad \limsup x_n = \lim_{N\to\infty} \underbrace{\sup\{x_n : n \geq N\}}_{\text{decreasing seq. } (\geq) \text{ bounded below by } -M. \text{ Converges.}}$

$-M \leq x_n \leq M$

$\liminf x_n = \lim_{N\to\infty} \underbrace{\inf\{x_n : n \geq N\}}_{\text{increasing seq } (\leq) \text{ bounded above by } M. \text{ Converges.}}$

Fact: A bounded seq x_n converges if and only if

$$\liminf x_n = \limsup x_n$$

both = $\lim x_n$ in this case.

Hadamard's formula: The radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is determined by

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}.$$

Def^n (x_n) is a Cauchy seq means: given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$|x_n - x_m| < \varepsilon$ when n and m are both larger than or equal to N $\quad (n, m \geq N)$.

Feeling: if terms of seq get closer and closer, they must get close to <u>something</u> (the limit).

<u>Easy fact</u> If (x_n) converges, then it is a Cauchy seq.

<u>Why</u>: $x_n \longrightarrow L$. Let $\varepsilon > 0$. There is an N such that $|x_n - L| < \frac{\varepsilon}{2}$ when $n \geq N$.

Aha! $$|x_n - x_m| = |x_n - L + L - x_m|$$
$$\leq \underbrace{|x_n - L|}_{< \frac{\varepsilon}{2}} + \underbrace{|x_m - L|}_{\frac{\varepsilon}{2}} \quad \text{when } n, m \geq N.$$

<u>Easy fact</u> Cauchy seq are <u>bounded</u>.

Why: Take $\varepsilon = 1$. Get N such that

$$|x_n - x_m| < 1 \quad \text{when } n, m \geq N.$$

$$|x_N - x_m| < 1 \quad \text{when } m \geq N.$$

$$|x_m| = |x_m - x_N + x_N| \leq \underbrace{|x_m - x_N|}_{< 1 \text{ when } m \geq N} + |x_N| < |x_N| + 1$$

Aha! $M = \text{Max}(|x_1|, |x_2|, \ldots, |x_N|, |x_N| + 1)$ is a <u>bound</u>.

More interesting: <u>Cauchy criterion</u>: Cauchy sequences converge.

<u>Why</u> Suppose (x_n) is a Cauchy seq. Then (x_n) is <u>bounded</u>.

Bolzano-Weierstraß $\Rightarrow$ (x_n) has a conv subseq (x_{n_k}). Say $x_{n_k} \to L$.

<u>Claim</u>: $x_n \to L$ too.

<u>Why</u>: Let $\varepsilon > 0$. There is an N such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \text{when} \quad n, m \geq N.$$

[Hmmm. $$|x_n - L| = |x_n - x_{n_k} + x_{n_k} - L|$$
$$\leq \underbrace{|x_n - x_{n_k}|}_{\text{Need this } < \frac{\varepsilon}{2}} + \underbrace{|x_{n_k} - L|}_{\text{Need } < \frac{\varepsilon}{2}}.$$]

Since $x_{n_k} \to L$, there is a N_2 such that

$$|x_{n_k} - L| < \frac{\varepsilon}{2} \quad \text{when} \quad k \geq N_2.$$

Can always choose an n_k far out so that $k \geq N_2$ and $n_k \geq N$. The idea above works! ✓

Fascinating fact: Cauchy criterion $\Longleftrightarrow$ least upper bound principle.

Dedekind: Dedekind cuts of $\mathbb{Q} \approx \mathbb{R}$. "Model" that make axioms of $\mathbb{R}$ (including l.u.b. princ) hold.

Another model: Cauchy seq $(x_n), (y_n)$. Equivalence $\sim$

$$\text{Say } (x_n) \sim (y_n) \text{ means } \lim_{n\to\infty} |x_n - y_n| = 0.$$

Equivalence class $[(x_n)] = \{(y_n) \text{ Cauchy} : (y_n) \sim (x_n)\}$.

$\mathbb{R} \approx \{[(x_n)] : (x_n) \text{ Cauchy seq}\}$.

<u>EX</u> $x_1 = 1$, $x_2 = 2$. Starting with $n=3$:

$$x_n = \frac{x_{n-1} + x_{n-2}}{2} = \text{ave of previous } 2.$$

Claim: (x_n) is Cauchy.

<u>Step 1</u>: $0 < x_n \le 2$ for all n. (Why?)

Induction. $x_1, x_2 \le 2$. Assume we know

$x_1, x_2, \ldots, x_N$ are all in this range.

$$x_{N+1} = \frac{x_N + x_{N-1}}{2} \longleftarrow \text{midpoint in range. (or use } \Delta \text{ ineq.)}$$

Step 2 $\quad |x_n - x_{n+1}| = \frac{1}{2^{n-1}}$. (Book: show this)

True for $n=1$. Suppose we know $|x_n - x_{n+1}| = \frac{1}{2^{n-1}}$.

hate this!

$$|x_{n+1} - x_{n+2}| = \left| \underbrace{\tfrac{1}{2}(x_n + x_{n-1})}_{= x_{n+1}} - \tfrac{1}{2}(x_{n+1} + x_n) \right|$$

$$= \left| \tfrac{1}{2}(x_{n+1} - x_n) \right|$$

$$= \tfrac{1}{2} |x_n - x_{n+1}| = \tfrac{1}{2} \cdot \tfrac{1}{2^{n-1}} = \frac{1}{2^{(n+1)-1}} \checkmark$$

Ind. hyp. $\Rightarrow$ true for $n+1$. True for all n.

Step 3 : Show (x_n) is Cauchy.

Say $m > n$.

$$|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + \cdots + - x_{m-1} + x_{m-1} - x_m|$$

$$\le \underbrace{|x_n - x_{n+1}|}_{\frac{1}{2^{n-1}}} + \underbrace{|x_{n+1} - x_{n+2}|}_{\frac{1}{2^n}} + \cdots + \underbrace{|x_{m-1} - x_m|}_{\frac{1}{2^{(m-1)-1}}}$$

$$\frac{1}{2^{n-1}} \left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n-1}} \right]$$

Geom series $\quad 1+r+r^2+\cdots+r^N = \dfrac{1-r^{N+1}}{1-r}$

So $\quad |x_n - x_m| < \underbrace{\dfrac{1}{2^{n-1}} \left(\dfrac{1-\left(\frac{1}{2}\right)^{m-n}}{1-\frac{1}{2}} \right)}$

$$2\,\frac{1}{2^{n-1}}\left(1-\left(\frac{1}{2}\right)^{m-n}\right) < 2\cdot\frac{1}{2^{n-1}} = \frac{1}{2^{n-2}}$$

Can make $<$ any $\varepsilon > 0$. $\quad$ Cauchy!

What does it converge to? $\quad$ Hmmm.

$$x_{n+1} = \frac{x_n + x_{n-1}}{2}$$

$$\downarrow \qquad\qquad \downarrow$$

$$L = \frac{L+L}{2}$$ Darn! $L=L$.

Book: $L = \frac{5}{3}$.