Lesson 15 Cauchy criterion, part 2 (3.5)

Seq (x_n) is Cauchy if, given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ when $n, m \geq N$.

Cauchy criterion: (x_n) Cauchy $\Rightarrow$ x_n converges. ($\Leftarrow$ easy).

Last time: showed $x_1 = 1$, $x_2 = 2$, $x_{n+1} = \dfrac{x_n + x_{n-1}}{2}$, $n \geq 2$ is Cauchy.

What is limit? (Cauchy $\Rightarrow$ L exists).

Book; Induction: Odd terms: $x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-1}}$

$$= 1 + \frac{1}{2}\Big(1 + \underbrace{\frac{1}{2^2}}_{\frac{1}{(2^2)^1}} + \underbrace{\frac{1}{2^4}}_{\frac{1}{(2^2)^2}} + \cdots + \underbrace{\frac{1}{2^{2n-2}}}_{\frac{1}{(2^2)^{n-1}}}\Big)$$

$$= 1 + \frac{1}{2}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^{n-1}\right)$$

$$= 1 + \frac{1}{2}\left[\frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}\right]$$

$$1 + r + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r}$$

$$\longrightarrow 1 + \frac{1}{2}\left[\frac{1 - 0}{1 - 1/4}\right] = \frac{5}{3}$$

Book: So $L = \frac{5}{3}$ (Why?) We know Cauchy $\Rightarrow$ converge. We also know every subseq of a conv seq converges to limit L. So $L = \frac{5}{3}$.

<u>Trick</u>: $|x_n - x_m| \le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$

allows us to prove:

<u>Def^n</u>: A seq (x_n) is called "contractive" if

$$|x_{n+2} - x_{n+1}| \le C\,|x_{n+1} - x_n| \quad \text{for all } n \in \mathbb{N}$$

where $0 < C < 1$.

Consequently $|x_{n+2} - x_{n+1}| \le C\,\underbrace{|x_{n+1} - x_n|}_{\le C\,\underbrace{|x_n - x_{n-1}|}_{\le C\,|x_{n-1} - x_{n-2}|\ \cdots}}$

$$\cdots \le C^n\,|x_2 - x_1|$$

Aha! Compare $|x_n - x_m|$ to

$$\underbrace{\left(C^{n-2} + C^{n-3} + \cdots + C^{m-1}\right)}_{C^{n-2}\left[\text{Geom series}\right]}\,|x_2 - x_1|$$

Can make small exactly as in example.

<u>Thm</u> Every contractive seq must converge.

(Because they are Cauchy)

Fibonacci #'s and the Golden Ratio $\left(\frac{-1+\sqrt{5}}{2}\right)$

(Nautilus picture here)

$f_1, f_2 = 1$. Starting with $n=2$: $f_{n+1} = f_n + f_{n-1}$

Let $x_n = \dfrac{f_n}{f_{n+1}}$.

Hmmm. $$x_{n+1} = \frac{f_{n+1}}{f_{n+2}} = \frac{f_{n+1}}{f_{n+1}+f_n} \cdot \frac{\left(\frac{1}{f_{n+1}}\right)}{\left(\frac{1}{f_{n+1}}\right)}$$

$$= \frac{1}{1+\frac{f_n}{f_{n+1}}} = \frac{1}{1+x_n}$$

$$\boxed{x_{n+1} = \frac{1}{1+x_n}}$$ ← recursion formula

Step 0: Induction: $\frac{1}{2} \le x_n \le 1$ (easy)

$$\frac{1}{2}+1 \le 1+x_n \le 1+1$$

$$\boxed{\underline{\frac{2}{3} \ge \frac{1}{1+x_n}} \ge \frac{1}{2}}$$

Claim: (x_n) is contractive.

$$|x_{n+1} - x_n| = \left| \frac{1}{1+x_n} - \frac{1}{1+x_{n-1}} \right|$$

$$= \left| \frac{(1+x_{n-1}) - (1+x_n)}{(1+x_n)(1+x_{n-1})} \right|$$

$$= \frac{|x_n - x_{n-1}|}{(1+x_n)(1+x_{n-1})}$$

$$= \frac{1}{1+x_n} \cdot \frac{1}{1+x_{n-1}} \cdot |x_n - x_{n-1}| \leq \underbrace{\left(\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right)}_{\frac{4}{9} < 1!} |x_n - x_{n-1}|$$

✓

So (x_n) Cauchy. So $x_n \to L$. What is L?

Since $x_n \geq \frac{1}{2}$ for all n, $L \geq \frac{1}{2}$.

Trick:

$$x_{n+1} = \frac{1}{1+x_n}$$

$$\downarrow \qquad \downarrow$$

$$L = \frac{1}{1+L}$$

Great!

$$L^2 + L - 1 = 0$$

$$L = \frac{-1 \pm \sqrt{1+4}}{2} = \begin{cases} \frac{\sqrt{5}-1}{2} > 0 \\ \frac{-\sqrt{5}-1}{2} < 0 \leftarrow \text{Out!} \end{cases}$$

So $L = \frac{\sqrt{5}-1}{2}$.

Exam 1: In class on Wed, Feb. 28.

One two-sided standard sized paper hand-written only on both sides. Closed books, notes, no electronics.

Only need pencils & erasers.

Covering Chapters 1, 2, 3, 4.1 – 4.2

3.6 "Properly divergent sequences"

$$\lim_{n\to\infty} x_n = \begin{cases} +\infty \\ -\infty \end{cases}$$

Def^n: $\lim_{n\to\infty} x_n = \infty$ means, given $M>0$, no matter how big, there is an $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$.

EX: $\lim_{n\to\infty} f_n = \infty$ ← $\infty = +\infty$.

Def^n: $\lim_{n\to\infty} f_n = -\infty$ similar. (Reverse ineq. make M neg.)