Lesson 16 $\lim_{n\to\infty} x_n = \begin{cases} +\infty \\ -\infty \end{cases}$, Series. (3.6, 3.7)

p. 77 : 12d. $\left(1-\frac{1}{n}\right)^n \to ?$ Know $\left(1+\frac{1}{n}\right)^n$ increasing seq bounded above by 3, so it converges to, by def^n, e.

$2 < e < 3$. You can assume that you know $\left(1-\frac{1}{n}\right)^n \to L$ and $L > 0$. Then

$$\left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^n = \left(1-\frac{1}{n^2}\right)^n = \underbrace{\left[\left(1-\frac{1}{n^2}\right)^{n^2}\right]}_{\downarrow\ ?}{}^{1/n}$$

$n \to \infty$: $\quad e \cdot L$

looks like $L^{1/n} \to 1$

$\left(1-\frac{1}{n^2}\right)^{n^2} \to L$ because that's a subseq of $\left(1-\frac{1}{n}\right)^n$.

And subseq of conv seq converge (to same limit).

Squeeze theorem! There is an $N \in \mathbb{N}$ such that

$(\varepsilon = \frac{L}{2})$: $\quad \frac{L}{2} < \left(1-\frac{1}{n^2}\right)^{n^2} < \frac{3L}{2}$ when $n \geq N$

$$\left(\frac{L}{2}\right)^{1/n} < \left(1-\frac{1}{n^2}\right)^n < \left(\frac{3L}{2}\right)^{1/n}$$

$n \to \infty$: $\quad 1 \qquad\qquad 1$

Squeeze!
This $\to 1$ too.

Last time defined $\lim_{n\to\infty} x_n = +\infty$.

<u>Def^n</u> $\lim_{n\to\infty} x_n = -\infty$ means, given $M \in \mathbb{R}$, no matter how <u>negative</u>, there is an $N \in \mathbb{N}$ such that $x_n \le M$ when $n \ge N$.

<u>EX</u> Fibonacci #'s $\to +\infty$.

$$\begin{array}{cccccccc} 1, & 1, & 2, & 3, & 5, & 8, & \ldots \\ f_1, & f_2, & f_3, & f_4, & f_5, & f_6 \end{array}$$

Induction: All $f_n \ge 1$. ✓

Starting at $n=6$, $f_n > n$. Induction step: Suppose $f_n > n$.

Then $f_{n+1} = \underset{>n}{f_n} + \underset{\ge 1}{f_{n-1}} > (n+1)$ ✓

Given $M > 0$, Archimedean princ yields $N \in \mathbb{N}$ with $N > M$. So for $n \ge \mathrm{Max}(6, N)$, we have

$f_n > n \ge N > M$. ✓

Know $r^n \to 0$ as $n \to \infty$ when $0 \le r < 1$.

$c^{1/n} \to 1$ as $n \to \infty$ when $c > 0$.

Today: $r^n \to +\infty$ when $r > 1$.

[Fresh calc: $r^n = e^{\ln r^n} = e^{n \ln r} \to \infty$ as $n \to \infty$
because $\ln r > 0$ when $r > 1$.]

First pinciples demo: $r > 1$. So $r = 1 + b$, $b > 0$.

$$r^n = (1+b)^n > 1 + nb \longrightarrow \infty \quad \text{as } n \to \infty.$$

Bernoulli's ineq [$b > -1$ ✓]

Easy fact: Suppose x_n is an increasing seq ($\leq$).

We know that if x_n is bounded above by M,

then $\lim\limits_{n\to\infty} x_n = L = \sup \{x_n\}_1^\infty \leq M$.

So today, if x_n is not bounded above, then

$\lim\limits_{n\to\infty} x_n = +\infty$. Similar fact for decreasing seq.

Series $\quad S_N = 1 + r + r^2 + \cdots + r^N = \dfrac{1 - r^{N+1}}{1 - r}$

If $S_N \to L$ as $N \to \infty$, write

$$S_N = \sum_{n=0}^{N} r^n$$

$$L = \sum_{n=0}^{\infty} r^n$$

Limit of the series.

Case $0 \le r < 1$. $\quad S_N = \dfrac{1 - r^{N+1}}{1-r} \longrightarrow \dfrac{1-0}{1-r}$

So $\boxed{\sum_0^\infty r^n = \dfrac{1}{1-r} \quad \text{when } 0 \le r < 1.}$

Case $r = 1$: $\quad \underbrace{1 + 1 + 1 + \cdots + 1}_{N+1 \text{ times}} = N+1 \to \infty$ as $N \to \infty$.

Case $r > 1$: $\quad S_N > 1 + 1 + 1 + \cdots + 1 = N+1. \to \infty$ as $N \to \infty$.

So $\displaystyle\sum_{n=0}^\infty r^n = +\infty$ when $r \ge 1$.

<u>EX</u>: $\quad S_N = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{N!} = \displaystyle\sum_{n=0}^N \frac{1}{n!} \qquad 0! = 1.$

Note S_N is increasing. Is it bounded above?

Actually, S_N is Cauchy seq. So it converges because of that. [And so it <u>is</u> bounded above.]

Check: $M > N$. $\quad S_M - S_N = \dfrac{1}{(N+1)!} + \dfrac{1}{(N+2)!} + \cdots + \dfrac{1}{M!}$

Hmmm. $\quad \dfrac{1}{n!} = \dfrac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} < \underbrace{\dfrac{1}{2} \cdot \dfrac{1}{2} \cdots \dfrac{1}{2}}_{n-1}$

$=2 \quad >2 \quad >2 \quad >2$

So $\frac{1}{n!} < \frac{1}{2^{n-1}}$ and we get

$$S_M - S_N < \underbrace{\underbrace{\underbrace{\underbrace{\frac{1}{2^{(N+1)-1}} + \frac{1}{2^{(N+2)-1}} + \cdots + \frac{1}{2^{M-1}}}_{\frac{1}{2^N} + \frac{1}{2^{N-1}} + \cdots + \frac{1}{2^{M-1}}}}_{\frac{1}{2^N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{M-N-1}}\right)}}_{\frac{1-\left(\frac{1}{2}\right)^{M-N}}{1-\frac{1}{2}}}}_{\frac{1}{2^{N-1}}\left(1-\left(\frac{1}{2}\right)^{M-N}\right) < \frac{1}{2^{N-1}}}$$

can make $<$ any given $\varepsilon > 0$.

So S_N is Cauchy. ✓

Fun fact: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$ too.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1}$$

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

S_N's Cauchy by same calculation as above $|S_M - S_N| \le$ same pos. terms.

Famous fact: The n-th term test.

If $\sum_{n=0}^{\infty} x_n$ converges, then $x_n \to 0$ as $n \to \infty$.

One way test: If $x_n \not\to 0$ as $n \to \infty$, then the series $\sum_0^{\infty} x_n$ <u>diverges</u>.

<u>Note</u>: Only given that $x_n \to 0$ as $n \to \infty$, <u>can't</u> conclude that $\sum_0^{\infty} x_n$ converges.

<u>Pf</u>: $$S_N - S_{N-1} = \sum_0^{N} x_n - \sum_0^{N-1} x_n = x_N$$

$N \to \infty$: $L - L$. So $x_N \to L - L = 0$ as $N \to \infty$.

Famous examples: p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

$$\sum_1^{\infty} \frac{1}{n} = +\infty$$

$$\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$