# Lesson 17 Series, cont'd (3.7, 4.1)

Series of non-negative terms $x_n \geq 0$. $S_N = \sum_{n=1}^{N} x_n$ is an increasing seq. If it is bounded (above), then it converges.
So either $\sum_{n=1}^{\infty} x_n = +\infty$ or $\sum_{1}^{\infty} x_n = L$ exists.

EX $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \qquad \text{mult by } n(n+1) =$$

$$0\cdot n + 1 = A(n+1) + Bn = (A+B)n + A$$

$A+B=0 \quad B=-A=-1$
$A=1$ ✓

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \longrightarrow 1 - 0 \text{ as } n \to \infty.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Problem: $\displaystyle\frac{1}{n(n+1)} \sim \frac{1}{n^2}$

Hmmm: $\displaystyle\frac{1}{(n+1)(n+1)} < \frac{1}{n(n+1)}$

$$\underbrace{1 + \frac{1}{2^2} + \cdots + \frac{1}{(n+1)^2}}_{<} < 2$$

$$\downarrow$$

$$1 \quad \frac{1}{1\cdot 2} + \cdots + \frac{1}{n(n+1)}$$

Bndd above.
Pos terms.
So converges.

<u>EX</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u> even though terms $\to 0$.

<u>Why</u>: Suppose $S' = \sum_{1}^{\infty} \frac{1}{n}$. Then

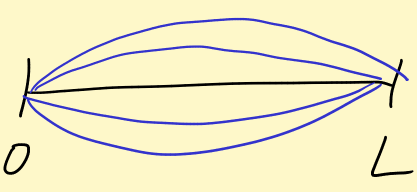
$$S' = \underbrace{\left(\frac{1}{1} + \frac{1}{2}\right)} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)} + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{n} + \frac{1}{n+1}\right) + \cdots$$

$$> \underbrace{\underbrace{\left(\frac{1}{2} + \frac{1}{2}\right)}_{1} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{6} + \frac{1}{6}\right)}_{\frac{1}{3}} + \cdots}_{S'}$$

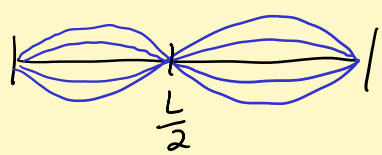
Deduce $S' > S'$ ! Assumption $S'$ exists must be <u>false</u>.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is the <u>harmonic</u> series.

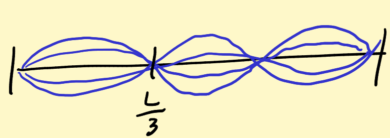
Vibrating string



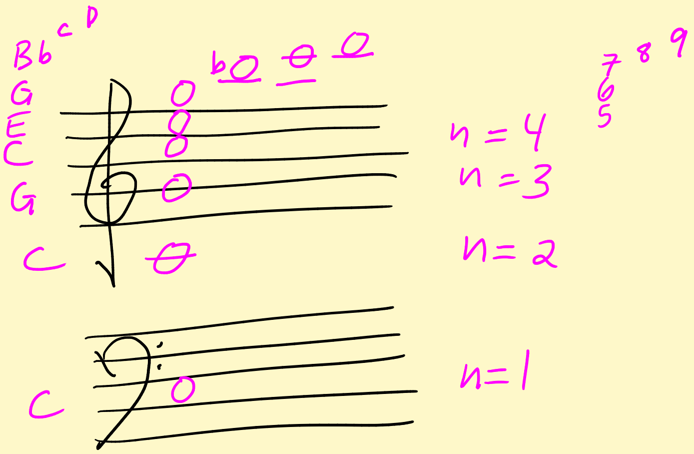

pitch




octave higher




octave & a fifth




Book: $S_N = \sum_1^N \frac{1}{n}$ is a very slow growing sequence, but it grows forever!

Comparison test $(x_n), (y_n)$ seq

$$0 \le x_n \le y_n$$

1) $\sum_1^\infty y_n$ conv $\Longrightarrow$ $\sum_1^\infty x_n$ conv

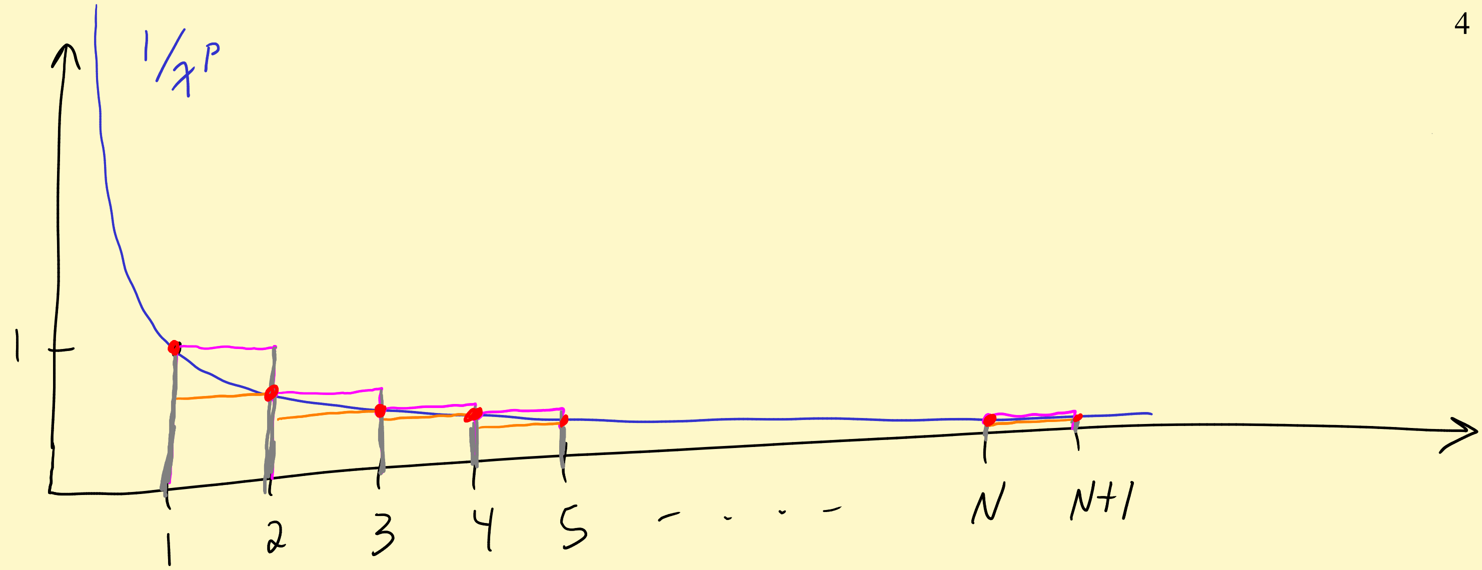
and $\sum_1^\infty x_n \le \sum_1^\infty y_n$.

2) $\sum_1^\infty x_n$ diverges $\Longrightarrow$ $\sum_1^\infty y_n$ diverges too.

P-Series $\sum_1^\infty \frac{1}{n^p}$ $\begin{cases} \text{Converges when } p > 1 \\ \text{Diverges when } 0 \le p \le 1 \end{cases}$

Ex: So $\sum_1^\infty \frac{1}{\sqrt{n}}$ diverges, but $\sum_1^\infty \frac{1}{n^{1.0001}}$ converges.

Freshman calculus Integral test!




$$\frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(N+1)^p} < \int_1^{N+1} \frac{1}{x^p}\,dx < \frac{1}{1^p} + \cdots + \frac{1}{N^p}$$

left side bounded. So convergent $\Longleftarrow$ Bounded as $N \to \infty$

Unbounded, $N \to \infty \implies$ right side $\to \infty$ too.

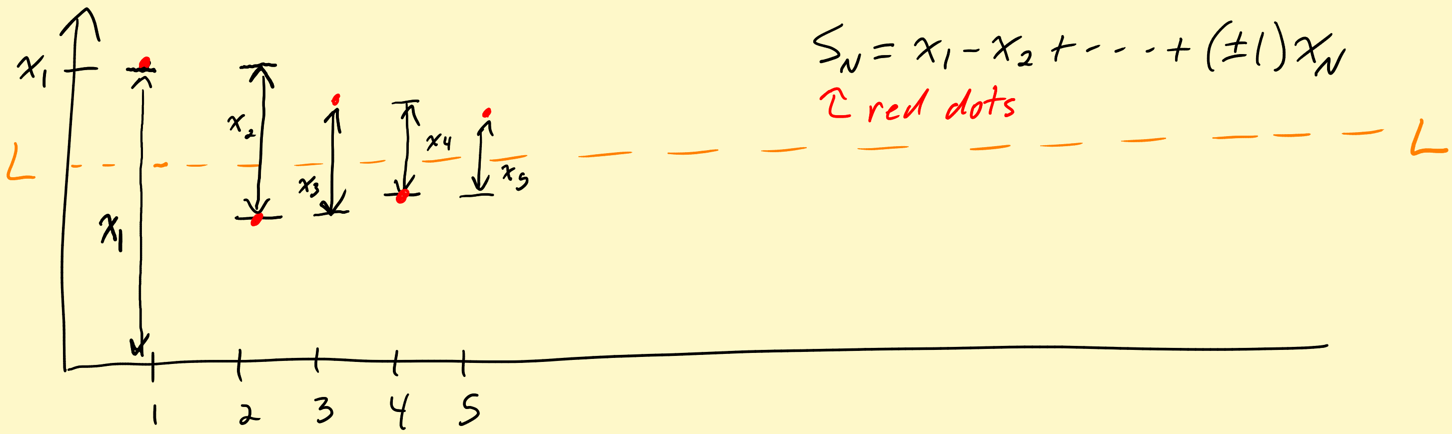
$$p \neq 1: \quad \int_1^{N+1} x^{-p}\,dx = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^{N+1} \qquad \begin{matrix} -p+1 > 0 \text{ bad.} \\ -p+1 < 0 \text{ good.} \end{matrix}$$

$$p = 1: \quad \int_1^{N+1} \frac{1}{x}\,dx = \ln(N+1) \to \infty \quad \text{as } N \to \infty.$$

Fun fact: $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges!

<u>Alternating series test</u> $x_n > 0$, $x_n$ decreasing, and $x_n \to 0$. Then $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ converges.

$$L = x_1 - x_2 + x_3 - x_4 + \cdots$$




$S_N = x_1 - x_2 + \cdots + (\pm 1) x_N$

red dots

Odd terms $S_{2n-1}$ are decreasing and bounded below by $x_1 - x_2$.

Even terms $S_{2n}$ are increasing and bounded above by $x_1$.

$$\text{So} \quad \left.\begin{array}{l} S_{2n-1} \to L_o \\ S_{2n} \to L_e \end{array}\right\} \text{Same!}$$

$$S_{2n+1} - S_{2n} = x_{2n+1}$$

$$L_o - L_e = 0 \qquad \text{So } L_o = L_e. = L$$

$$\text{So} \quad \sum_1^\infty (-1)^{n+1} x_n = L.$$

Limit comparison test $\quad (x_n), (y_n)$

$x_n > 0, \quad y_n > 0$

Suppose $r = \lim_{n\to\infty} \frac{x_n}{y_n}$ exists.

Case $r > 0$: $\sum_1^\infty x_n$ conv $\Longleftrightarrow$ $\sum_1^\infty y_n$ conv.

Case $r = 0$: $\sum y_n$ conv $\Longrightarrow$ $\sum_1^\infty x_n$ conv