Lesson 18 Limits of functions (4.1) (Practice problems posted)

Limit comparison theorem $(x_n), (y_n)$ sequences of positive terms.

$\lim\limits_{n\to\infty} \frac{x_n}{y_n}$ exists $= r.$ $[r \geq 0$ since $x_n > 0, y_n > 0]$

Case $r > 0$ $\sum_1^\infty x_n$ and $\sum_1^\infty y_n$ do same thing.

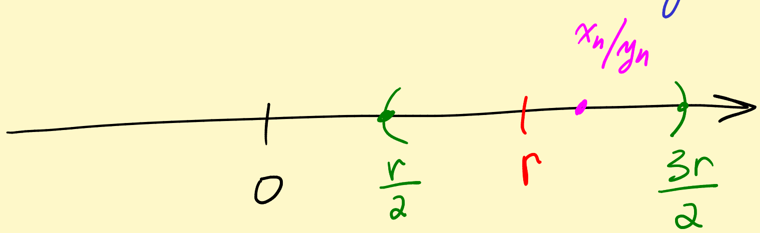
(either both converge or both diverge)

Case $r = 0$ $\sum_1^\infty y_n$ converges $\Rightarrow \sum_1^\infty x_n$ converges.

(so $\sum x_n$ diverges $\Rightarrow \sum y_n$ diverges too)

Why $\frac{x_n}{y_n} \to r$ Think $x_n \approx r y_n$

$r > 0$



$\varepsilon = \frac{r}{2}$

There is an $N \in \mathbb{N}$ such that $\left|\frac{x_n}{y_n} - r\right| < \underbrace{\frac{r}{2}}_{\varepsilon}$

when $n \geq N$.
$$-\frac{r}{2} < \frac{x_n}{y_n} - r < \frac{r}{2}$$

$$\frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2}$$

$\left(\frac{r}{2}\right) y_n < x_n$

Aha!

If $\sum x_n < \infty$ (series converges) $\Rightarrow \sum y_n < \infty$ converges ✓

$x_n < \left(\frac{3r}{2}\right) y_n$

$\sum y_n < \infty \Rightarrow$ same about $\sum x_n$.

Similar reasoning handles divergent case $\sum_1^\infty x_n = \sum_1^\infty y_n = \infty$.

Case $r=0$ Take $\varepsilon = 1$. $\frac{x_n}{y_n} < 1 \quad n \ge N$

$x_n < y_n \quad n \ge N$. Easier

only one ineq to use.

EX $\sum_1^\infty \frac{1}{\sqrt{n^2+n-1}}$ $\underset{\sim}{\text{Limit compare}}$ $\sum_1^\infty \frac{1}{n}$ both diverge.

$$\frac{\frac{1}{\sqrt{n^2+n-1}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2+n-1}} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}} \to \frac{1}{\sqrt{1+0+0}} = \underline{1}$$

$r=1$

EX $\sum \frac{1}{\sqrt{n^3+2n-1}} \sim \sum \frac{1}{n^{3/2}}$ $p = \frac{3}{2} > 1$. Both converges.

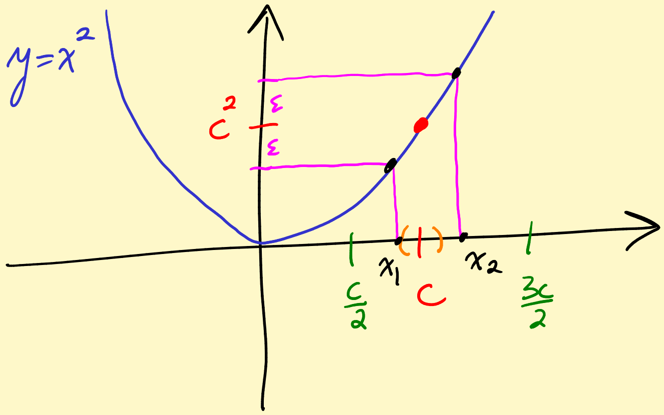
Limits of functions Newton, Leibniz

$$DQ = \underbrace{\frac{f(x)-f(a)}{x-a}}_{\text{not defined at } x=a.} \longrightarrow f'(a)$$

$\lim_{x\to a} f(x) = L$ means: Given $\varepsilon > 0$, no matter how small, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ when $|x-a| < \delta$ and $x \neq a$.

EX: Suppose $c > 0$. Show that $\lim_{x \to c} x^2 = c^2$.



Given $\varepsilon > 0$.

$|f(x) - c^2| =$

$|x^2 - c^2| < \varepsilon$ ← want

Hmmm. What's best possible δ?

$x_1^2 = c^2 - \varepsilon$ | $x_2^2 = c^2 + \varepsilon$

$x_1 = \sqrt{c^2 - \varepsilon}$ | $x_2 = \sqrt{c^2 + \varepsilon}$

Which is smaller, $c - \sqrt{c^2 - \varepsilon}$ or $\underbrace{\sqrt{c^2 + \varepsilon} - c}_{\text{(this one is smaller)}}$?

Ans = best δ.

Too much work! Just need a δ that works!

Hmmm. $|x^2 - c^2| = |(x-c)(x+c)|$

$$= \underbrace{|x-c|}_{<\delta} \underbrace{|x+c|}_{\text{need to bound this by some } M > 0.} \overset{\text{want}}{<} \varepsilon$$

Then $\delta = \frac{\varepsilon}{M}$ would work.

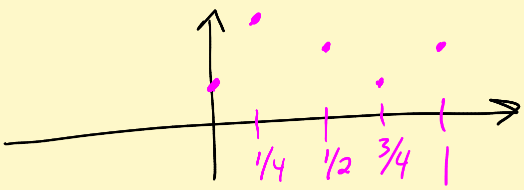
Hmmm. $\boxed{|x+c| \leq |x| + c}$. The "$\frac{c}{2}$" trick. Can start out taking $\delta < \frac{c}{2}$: $|x-c| < \frac{c}{2}$. Then

$\frac{c}{2} < x < \frac{3c}{2}$. So $|x+c| \le |x| + c < \underbrace{\frac{3c}{2}+c}_{\frac{5c}{2} = M}$

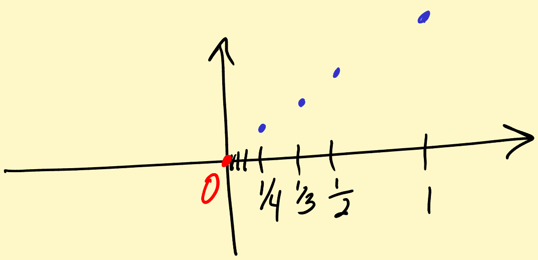
Need $\delta < \begin{cases} c/2 \\ \frac{\varepsilon}{M} = \frac{\varepsilon}{\left(\frac{5c}{2}\right)} \end{cases}$. Aha! Take $\delta = \min$ of those two.

Limits of fcns on <u>subsets</u> of $\mathbb{R}$

<u>EX</u>



Nothing resembling a limit here.



$\lim \frac{1}{n} = 0$

$\frac{1}{n}$'s "piling up" at $x = 0$. $x = 0$ is called an accumulation point of $S' = \{\frac{1}{n} : n \in \mathbb{N}\}$.

<u>EX</u>: $S' = \mathbb{Q}$ rational numbers.

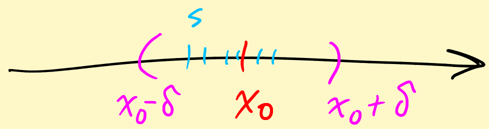
$\sqrt{2}$ is an accumulation pt of S'.

<u>Def^n</u>: $S' \subset \mathbb{R}$. x_0 is an acumulation pt of S', if, given any $\delta > 0$, no matter how small, there is a pt $s \in S'$ with $s \neq x_0$ such that $|s - x_0| < \delta$.

Easy check: x_0 is an acc. pt of S', then

$V_\delta(x_0)$ contains <u>infinitely</u> many $s \in S'$, $s \neq x_0$

for any $\delta > 0$.



Why. Take $\delta = 1$. Get $s_1 \neq x_0$. Take $\delta_2 = |s_1 - x_0|$. Get $s_2 \neq x_0$

Take $\delta_3 = |s_2 - x_0|$, etc. Get seq (s_n). ✓