Lesson 19 Limit theorems (4.2)

EX $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ when $c > 0$.

Let $\varepsilon > 0$. $\left(\sqrt{x} - \sqrt{c}\right) \cdot \dfrac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} = \dfrac{x - c}{\sqrt{x} + \sqrt{c}}$

So $\left|\sqrt{x} - \sqrt{c}\right| = \dfrac{|x-c|}{\sqrt{x}+\sqrt{c}}$ ← assume $x > 0$ so $\sqrt{x}$ makes sense

$< \dfrac{1}{\sqrt{c}} \underbrace{|x - c|}_{< \delta} \overset{\text{want}}{<} \varepsilon$

Want $\dfrac{\delta}{\sqrt{c}} < \varepsilon$. $\boxed{\delta < \varepsilon\sqrt{c}}$

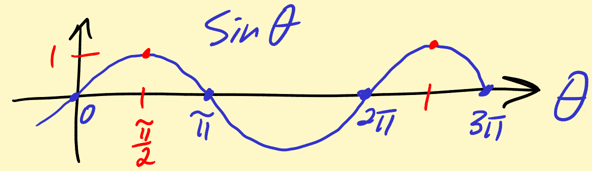
So, if $|x - c| < \varepsilon\sqrt{c}$, $x \neq c$, then $|\sqrt{x} - \sqrt{c}| < \varepsilon$. ✓

Hard thing: To say a limit DNE (does not exist) or is not $= L$, is a mouthfull because of all the quantifiers: all, every, there exists, ...

Handy fact: $f : (a,b) \to \mathbb{R}$. $c \in (a,b)$.

$\lim_{x \to c} f(x) = L \iff$ for every seq (x_n) where $x_n \in (a,b)$, $x_n \neq c$ all n, $\lim_{n \to \infty} f(x_n) = L$.

Fun examples $\sin \frac{1}{x}$

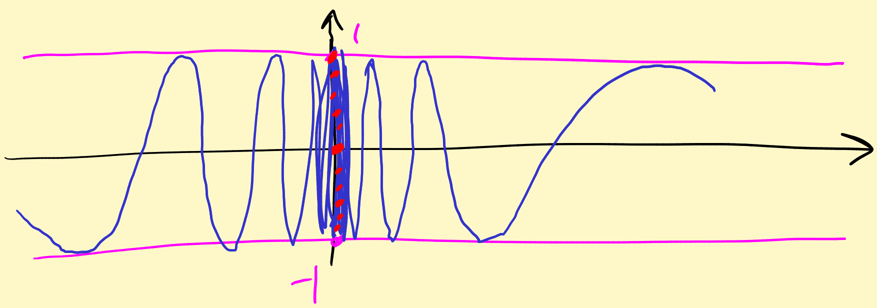


$\frac{1}{x} = n\pi$. $\quad x = \frac{1}{n\pi} = x_n \to 0$. $\sin \frac{1}{x_n} = \underline{\underline{0}}$.

$\frac{1}{x} = (2\pi)(n-1) + \frac{\pi}{2}$. $\quad x_n = \frac{1}{(2\pi)(n-1) + \frac{\pi}{2}} \to 0$ as $n \to \infty$.

$\sin \frac{1}{x_n} = \underline{\underline{1}}$. Different!

So $\lim_{x \to 0} \sin \frac{1}{x}$ DNE.



Squeeze theorem (for limits of fcns)

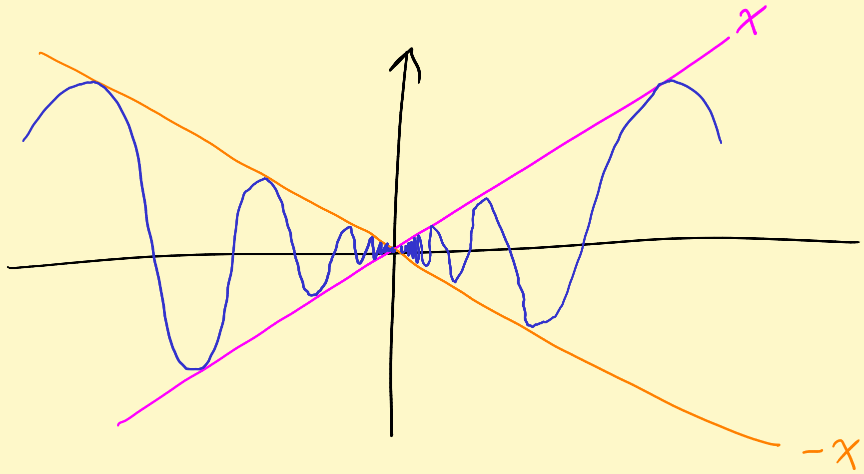
Suppose f, g, h on (a,b), $c \in (a,b)$ and

$$f(x) \le g(x) \le h(x) \quad \text{for } x \ne c.$$

If $f(x) \to L$, $h(x) \to L$ as $x \to c$,

then $\lim_{x \to c} g(x)$ exists and $= L$ too.

EX



Aha! $\quad -|x| \le x \sin \frac{1}{x} \le |x| \;, \quad x \neq 0.$

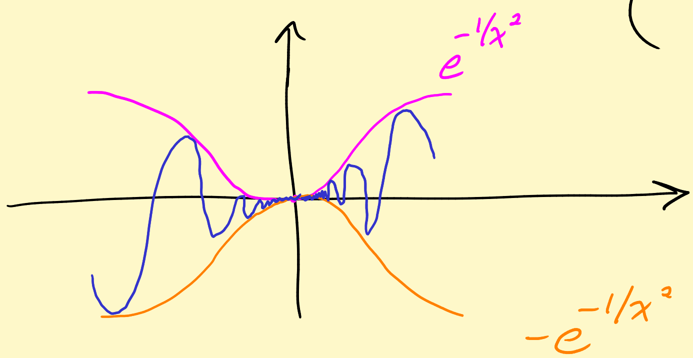
$x \to 0: \quad 0 \qquad\qquad 0$

Define

$$h(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then $\lim\limits_{x \to 0} h(x) = h(0) = 0.$ [h is "continuous" at $x = 0$

My favorite example in this genre. $\begin{cases} e^{-1/x^2} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



All the derivatives exist and are continuous, even at $x = 0$!

All derivatives vanish at $x=0$!

Taylor series $\equiv 0$, but fcn not $\equiv 0$.

Not every C^∞-smooth fcn can be expanded in a power series.

Divergence criteria: $\lim_{x\to c} f(x)$ DNE $\iff$

there is a seq (x_n) with $x_n \neq c$, $x_n \to c$ as $n \to \infty$

such that $\lim_{n\to\infty} f(x_n)$ DNE.

EX: $\sin\frac{1}{x}$. Above, got $x_n \to 0$ with $\sin\frac{1}{x_n} = 0$.

got $y_n \to 0$ with $\sin\frac{1}{y_n} = 1$.

Hmmm. Let $(z_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$

Limits of functions on sets like $\{\frac{1}{n} : n \in \mathbb{N}\}$, $\mathbb{Q}$ rationals.

Def^n If c is an accumulation pt of a set $S' \subset \mathbb{R}$,

and $f: S' \to \mathbb{R}$, then $\lim_{x\to c} f(x) = L$ means,

given $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(s) - L| < \varepsilon$ when $s \in S'$, $s \neq c$, and

$|s - c| < \delta$.

Remark All the facts we've studied carry over.

EX:
$$f(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational.} \end{cases}$$

Claim: $\lim_{x \to c} f(x)$ DNE at any $c \in \mathbb{R}$.

Hmmm. For $n \in \mathbb{N}$

$$c < q_n < c + \frac{1}{n}$$ Pick $q_n \in \mathbb{Q}$. Dense!

$$c < r_n < c + \frac{1}{n}$$ Pick $r_n \in \mathbb{R} - \mathbb{Q}$. Dense!

$q_n \neq c$, $q_n \to c$ as $n \to \infty$

$r_n \neq c$, $r_n \to c$ as $n \to \infty$

Merge them $(z_n) = (q_1, r_1, q_2, r_2, \ldots)$. See lim DNE.

Fancier divergence criterion. If you can find two subseq as above with $f(\text{subseq}) \to$ two different things, lim DNE.

Read easy facts in 4.2 Limit sum = Sum of limits etc.