Lesson 20 Limit theorems (4.2, 4.3)

Review Practice Probs on Monday

Fact A is a subset of $\mathbb{R}$ and c is a cluster point of A. If $\lim_{x \to c} f(x) = L$, then f is bounded near c, meaning there is a $\delta > 0$ and $M > 0$ such that $|f(s)| \le M$ when $|s-c| < \delta$ and $s \in A$, i.e., $|f|$ is bounded by M on $V_\delta(c) \cap A$.

Say "f is bounded on a neighborhood of c in A."

Why: Take $\varepsilon = 1$ in def^n of limit. Get $\delta > 0$ such that $|f(s) - L| < 1$ when $s \in A$, $|s-c| < \delta$, $s \ne c$.

$$-1 < f(s) - L < 1$$

$$L - 1 < f(s) < 1 + L$$

$$\text{So} \quad |f(s)| < \text{Max}(|L-1|, |1+L|) \overset{=}{\le} 1 + |L|$$

when $s \in A$, $s \neq c$.

only need when $c \in A$.

Take $M = \text{Max}(1 + |L|, \underbrace{|f(c)|})$. Works on $V_\delta(c) \cap A$.

Easy facts c cluster pt of $A \subset \mathbb{R}$. Suppose

$$\lim_{x \to c} f(x) = F, \quad \lim_{x \to c} g(x) = G.$$

Then
$$\lim_{x \to c} \left[f(x) + g(x) \right] \underset{\uparrow}{=} F + G$$

exists and =

$$\lim_{x \to c} kf(x) = kF, \quad k \text{ is a const.}$$

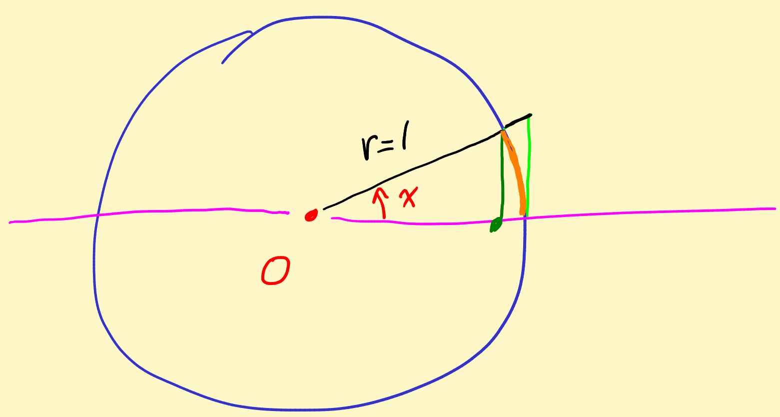
$$\lim_{x \to c} \left[k_1 f(x) + k_2 g(x) \right] = k_1 F + k_2 G$$

$$\lim_{x \to c} f(x) g(x) = FG$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided that } G \neq 0.$$

Famous example: $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1$

Freshman calculus



$$\left.\begin{array}{l} \text{Green line: length } \sin x \\ \text{Orange circular arc: length } x \text{ (in radians)} \end{array}\right] \sin x \leq x$$

More advanced fact $\boxed{x - \frac{x^3}{3!} \le \sin x \le x}$ when $x \ge 0$.

So $-\left(x - \frac{x^3}{3!}\right) \ge -\sin x \ge -x$ when $x \ge 0$.

$$(-x) - \frac{(-x)^3}{3!} \ge \sin(-x) \ge (-x)$$

Get $\boxed{x \le \sin x \le x - \frac{x^3}{3!}}$ when $x \le 0$

Book: $1 - \frac{1}{6}x^2 \le \frac{\sin x}{x} \le 1$ (why?)

when $x \ne 0$.

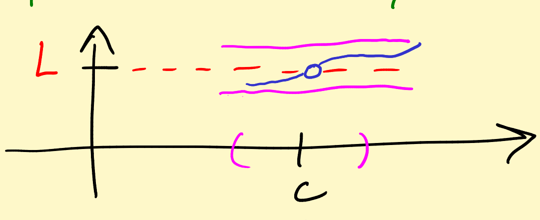
[Top box. Divide by x ← positive.
Bottom box. Divide by x ← negative. Flips ineq.

Aha! Squeeze thm yields $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Note: $\lim_{x \to 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0^2}{6} = 1$.

<u>Theorem</u> If $\lim_{x \to c} f(x) = L > 0$, then f must be <u>strictly</u> positive "nearby"

<u>Why</u>



Take $\varepsilon = \frac{L}{2}$

Get $|f(x)-L| < \frac{L}{2}$ when $|x-c|<\delta$, $x \neq c$

$$-\frac{L}{2} < f(x) - L < \frac{L}{2}$$

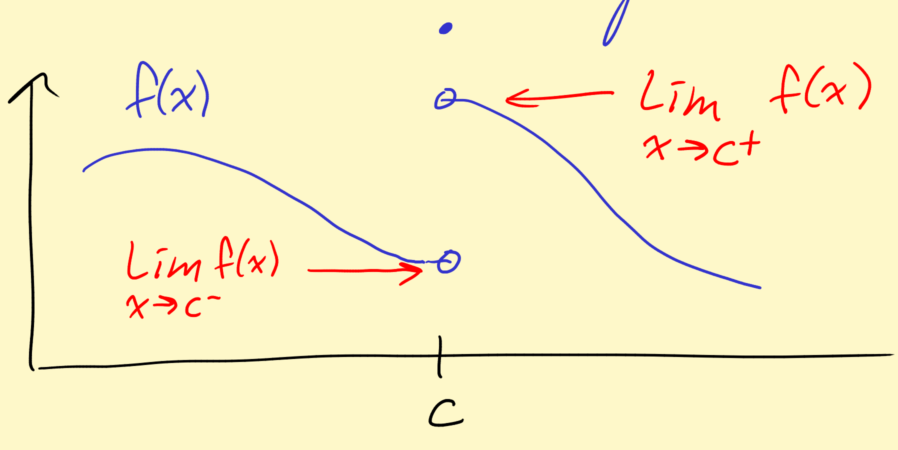
$$\frac{L}{2} < f(x) < \frac{3L}{2}$$

Great! See $f(x) > \frac{L}{2} > 0$ when $\underbrace{|x-c|<\delta,\ x\neq c.}_{x \in V_\delta(c) - \{c\}.}$

Same thing works with c a cluster pt of A, $f: A \to \mathbb{R}$.

$$x \in \left[A \cap V_\delta(c)\right] - \{c\}.$$

4.3 Left-hand and right-hand limits.



Def^{n}s: For any $\varepsilon > 0$ ---

$c < x < c + \delta$ $\quad c^+$

$c - \delta < x < c$ $\quad c^-$

Easy fact: $\lim_{x \to c} f(x) = L$

if and only if $\lim_{x \to c^{+,-}} f(x)$ exist

and are equal.

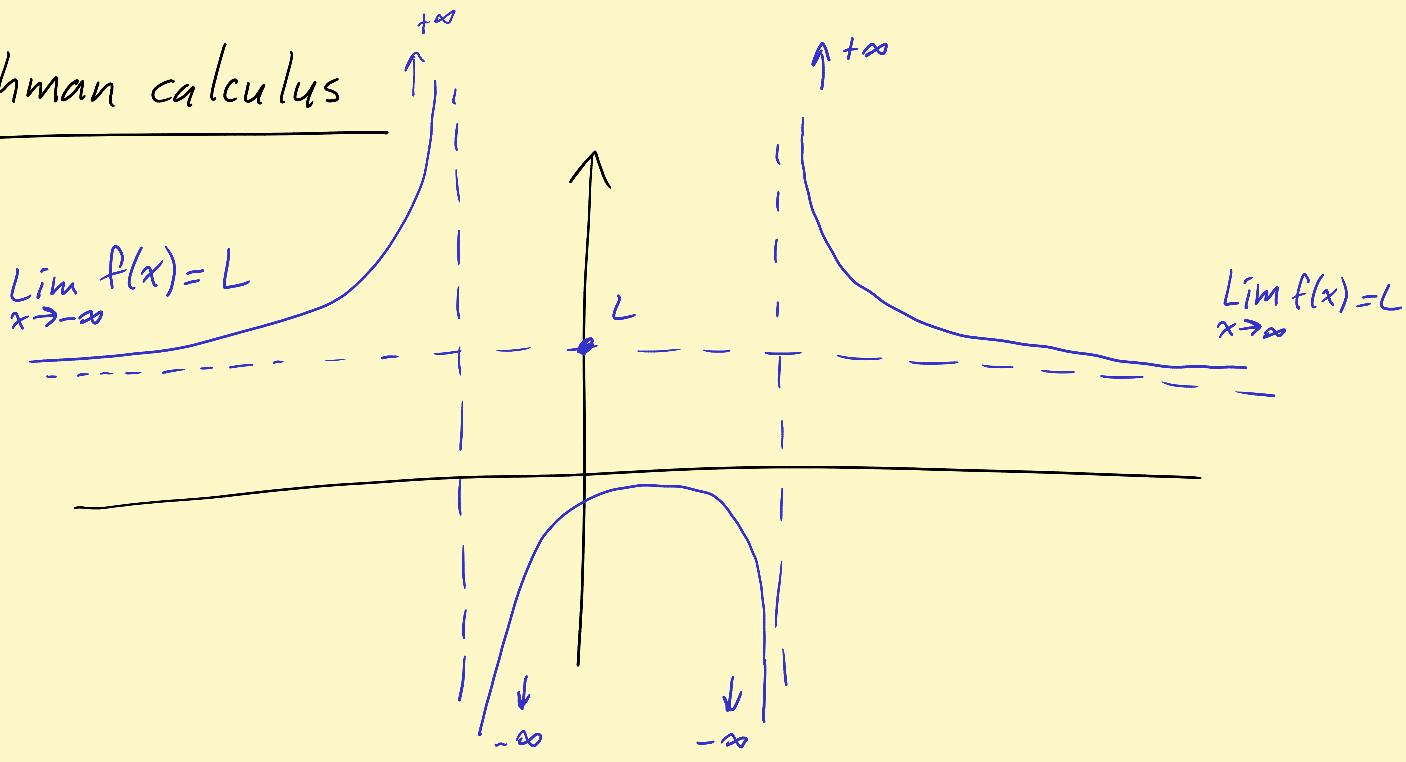
$\lim_{x \to c} f(x) = +\infty$ or $-\infty$.

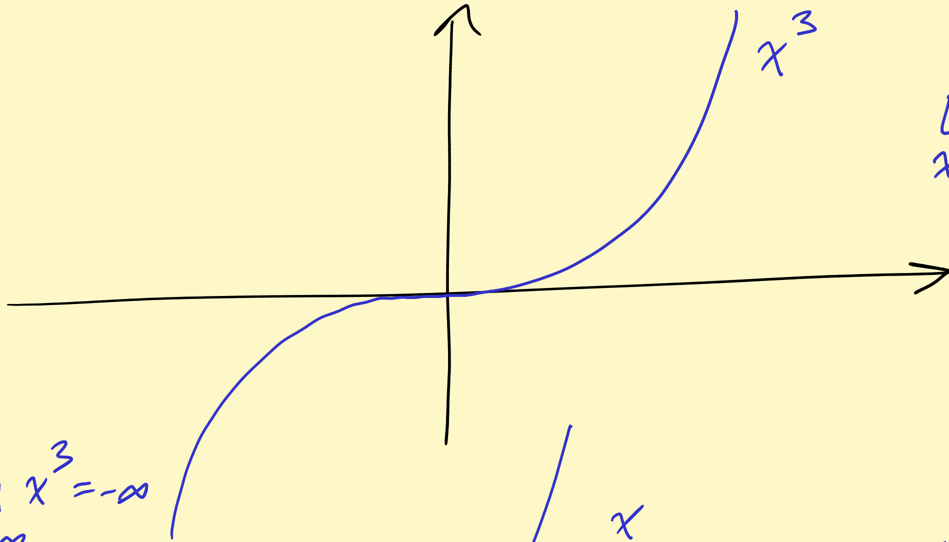
Def^n: Given $\begin{cases} M > 0, \text{ no matter how big, } \ldots \\ M < 0, \text{ no matter how negative, } \ldots \end{cases}$

there is a $\delta > 0$ ...

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} L \in \mathbb{R} \\ +\infty \\ -\infty \end{cases}$$

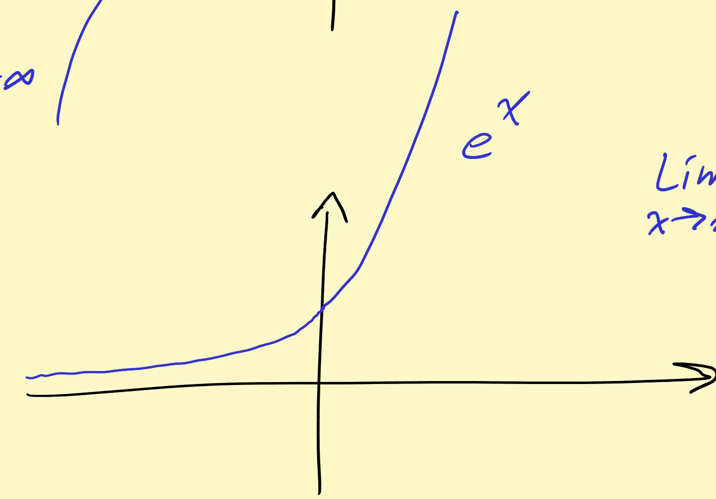
Freshman calculus





$\lim_{x \to \infty} x^3 = \infty$

$\lim_{x \to -\infty} x^3 = -\infty$



$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to -\infty} e^x = 0$

Let $M > 0$. Want $x^3 > M$. $(x > 0)$

$$x \geq M^{1/3}$$

So if $x > M^{1/3}$, then $x^3 > M$.

After Exam 1 in-class on Wed. 2/28, Start Chap 5, continuous fcns.