Review for Exam 1 (in-class, Wed., Feb 28, one two-sided hand-written crib sheet)

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

Suppose $x \in$ left side. Show x also in right.

Suppose $y \in$ right . . . . left.

p. 16 : 18 $\quad \underbrace{\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}}}_{S_n} > \sqrt{n} \quad n = 2, 3, \ldots$

$n = 2$ ✓

Suppose true for n: $S_n > \sqrt{n}$.

$$\text{Then} \quad S_{n+1} = S_n + \frac{1}{\sqrt{n+1}} > \underbrace{\sqrt{n} + \frac{1}{\sqrt{n+1}}}_{\overset{?}{>} \sqrt{n+1}}$$

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \overset{?}{>} \sqrt{n+1}$$

$$\frac{1}{\sqrt{n+1}} \overset{?}{>} \underbrace{\left(\sqrt{n+1} - \sqrt{n}\right) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}}_{\frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}}}$$

$(n+1) - n \leftarrow 1$

Yes!

Go backwards now.

p. 36 : 18a $\quad \max\{a, b\} = \frac{1}{2}\left(a + b + |a - b|\right)$

Trichotomy principle. $\begin{cases} a < b \\ a = b \\ a > b. \end{cases}$

Case $a = b$. ✓

Case $a < b$: $\quad Max\{a, b\} = b \overset{?}{=} \frac{1}{2}\left(a+b+\underbrace{[-(a-b)]}_{|a-b|}\right)$ ✓

etc.

Be able to define: Archimidean principle.

"Completeness" = "least upper bound property"

Density: $\mathbb{Q}$ is dense in $\mathbb{R}$. Given $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, there is a $q \in \mathbb{Q}$ with $r_1 < q < r_2$

$(\mathbb{R} - \mathbb{Q})$ dense in $\mathbb{R}$.

Supremum: least upper bound

Infinum: greatest lower bound

Cauchy sequence. Show (x_n) is a Cauchy seq =

Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{when} \quad n, m \geq N.$$

(Can make either n or m the bigger one.)

p. 39: 7 Suppose $A \subset \mathbb{R}$ is bound <u>above</u>. It <u>has</u> an upper bound M such that $a \leq M$ for all $a \in A$.

If c is an upper bound for A and <u>$c \in A$</u>, then $c = \operatorname{Sup} A$ = "least upper bound for A".

Suppose $\tilde{c} < c$ is also an upper bound.

Then $a \le \tilde{c} < c$ for all $a \in A$.

true for $a=c$ too because $c \in A$. See $c<c$. ✓.

So no such $\tilde{c}$. c is the <u>least</u> upper bound.

Know Cantor's proof that $[0,1]$ is uncountable using the Nested closed interval theorem.

Put $1+r+r^2+\cdots+r^N = \dfrac{1-r^{N+1}}{1-r}$ $(r \ne 1)$ on crib sheet.

<u>EX</u>: $\left(\frac{1}{2}\right)^{\overset{2\cdot 3}{6}} + \left(\frac{1}{2}\right)^{\overset{2\cdot 4}{8}} + \left(\frac{1}{2}\right)^{\overset{2\cdot 5}{10}} + \cdots + \left(\frac{1}{2}\right)^{2N}$

$$\left[\left(\tfrac{1}{2}\right)^2\right]^3 + \left[\left(\tfrac{1}{2}\right)^2\right]^4 + \cdots$$

$$\left(\tfrac{1}{4}\right)^3 + \left(\tfrac{1}{4}\right)^4 + \left(\tfrac{1}{4}\right)^5 + \cdots + \left(\tfrac{1}{4}\right)^N$$

$$\left(\tfrac{1}{4}\right)^3 \underbrace{\left[1 + \left(\tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{4}\right)^{N-3}\right]}_{\dfrac{1-\left(\frac{1}{4}\right)^{(N-3)+1}}{1-\frac{1}{4}}}$$

$$\frac{1-\left(\frac{1}{4}\right)^{(N-3)+1}}{1-\frac{1}{4}} \to \frac{1-0}{\frac{3}{4}} = \frac{4}{3}$$

as $N \to \infty$.

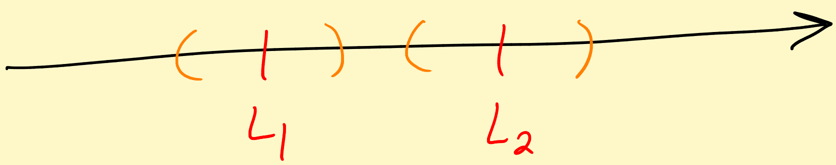
Important limits: $0 < r < 1$: $\lim_{n\to\infty} r^n = 0$

$0 < c$: $\lim_{n\to\infty} c^{1/n} = 1$

$$\lim_{n\to\infty} n^{1/n} = 1$$

$$\lim_{n\to\infty} \left(n + \frac{1}{n}\right)^n = e \quad \text{where} \quad 2 < e < 3.$$

Prove that a seq can't have <u>two</u> limits. (x_n)



Aha! Take $\varepsilon < \frac{L_2 - L_1}{2}$. Get N_1 for L_1.
Get N_2 for L_2.

Oops! For $N = \text{Max}(N_1, N_2)$. Get $\lightning$.

Know $\lim_{n\to\infty} \frac{1}{n} = 0$.

$\lim_{n\to\infty} \frac{2^n}{n!} = ?$

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdot \cdots 2}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}$$

$$= 2 \cdot 1 \cdot \underbrace{\left(\frac{2}{3}\right)}_{<1}\underbrace{\left(\frac{2}{4}\right)}_{<1} \cdots \underbrace{\left(\frac{2}{n-1}\right)}_{<1} \cdot \left(\frac{2}{n}\right)$$

$$< 2 \cdot \frac{2}{n}$$

$x_n = 0 \qquad y_n = \frac{2^n}{n!} \qquad z_n = \frac{4}{n}$

$$0 \le \frac{2^n}{n!} < \frac{4}{n}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

Squeeze thm $\Rightarrow$ $\frac{2^n}{n!} \to 0$ as $n \to \infty$.

<u>EX</u>: $n^{1/n^2} = \left[n^{1/n} \right]^{1/n}$

$n^{1/n} \longrightarrow 1$ as $n \to \infty$. So there is a $N \in \mathbb{N}$ such that $n^{1/n} < 2$ when $n \ge N$.

$$1 < n^{1/n} < 2 \quad \text{when } n \ge N.$$

So $$1^{1/n} < \left[n^{1/n} \right]^{1/n} < 2^{1/n} \quad \text{when } n \ge N.$$

$$\downarrow \qquad\qquad \downarrow$$

$$1 \qquad \text{Squeeze thm.} \qquad 1$$

p. 77: 7 $\qquad x_1 = a > 0. \qquad \underbrace{x_{n+1} = x_n + \frac{1}{x_n}}$

Does $x_n \to L$?

$$\boxed{x_{n+1} - x_n = \frac{1}{x_n}}$$

Step 1. Show $x_n \geq a$ for all n.

$n=1$ ✓

Assume true for n: $x_n \geq a$.

Then $x_{n+1} = x_n + \frac{1}{x_n} \geq a + \frac{1}{x_n} > a$ ✓

Step 2. Seq is increasing.

$x_{n+1} - x_n = \frac{1}{x_n} > 0.$ ✓

Know: an increasing seq that is bndd above converges.

Hmmm. If limit L exists, then $L \geq a > 0$.

$$x_{n+1} = x_n + \frac{1}{x_n}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$L = L + \frac{1}{L}$$

$$0 = \frac{1}{L}$$

↯. Crazy.

So L doesn't exist. So $\lim_{n\to\infty} x_n = +\infty$.

p-Series. P. 91: 5. $x_n = \sqrt{n}$ is not Cauchy even though $|x_n - x_{n+1}| \to 0$ as $n \to \infty$.