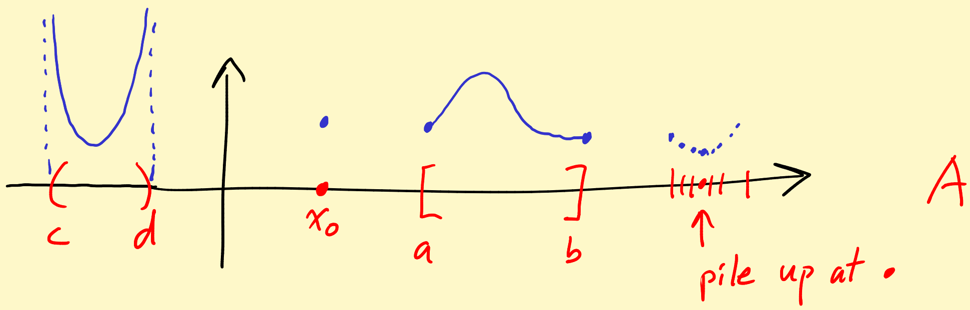
Lecture 21 Continuous functions (5.1)

f is <u>continuous</u> at c if $\lim_{x\to c} f(x) = f(c)$,

meaning that, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ if $|x - c| < \delta$.

What if $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$?



x_0 is an "isolated" pt of A : not a cluster pt.

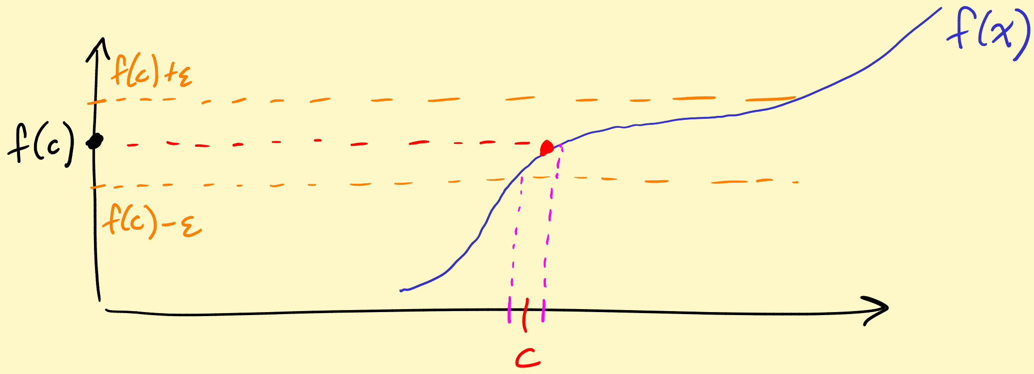
$f(x_0)$ could anything. f is cont at x_0.

If $c \in A$ is a cluster pt of A, then continuity at c means, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - f(c)| < \varepsilon$ when $|x - c| < \delta$ <u>and $x \in A$</u>.

<u>Def^n</u> $f : A \to \mathbb{R}$ is <u>continuous</u> on A means f is continuous at each pt in A.

Geometric meaning : Think of ε as an "error" or "tolerance".



"Topological way of thinking": Given $\varepsilon>0$, there is a $\delta>0$ such $f(V_\delta(c)) \subset V_\varepsilon(f(c))$.

Wild function: Thomae's fcn. $T:[0,1]\to\mathbb{R}$

Let $T(x)=0$ if x is irrational.

If $x=\frac{p}{q}$ is rational, can write x as

$x=\frac{\tilde{p}}{\tilde{q}}$ where $\tilde{p},\tilde{q}$ are positive integers with no common factors (relatively prime).

Then $T(x)=\frac{1}{\tilde{q}}$. [Let $T(0)=1$, $T(1)=1$.]

Cool thing: T is continuous at every irrational pt in $[0,1]$, discontinuous at rational pts.

See p. 127 for a graph of $T(x)$.

Sequential criterion for continuity $f:A\to\mathbb{R}$ is continuous at $c\in A$ if and only if $\lim_{n\to\infty} f(x_n)=f(c)$ for

<u>every</u> sequence (x_n) with $x_n \in A$ for all n and

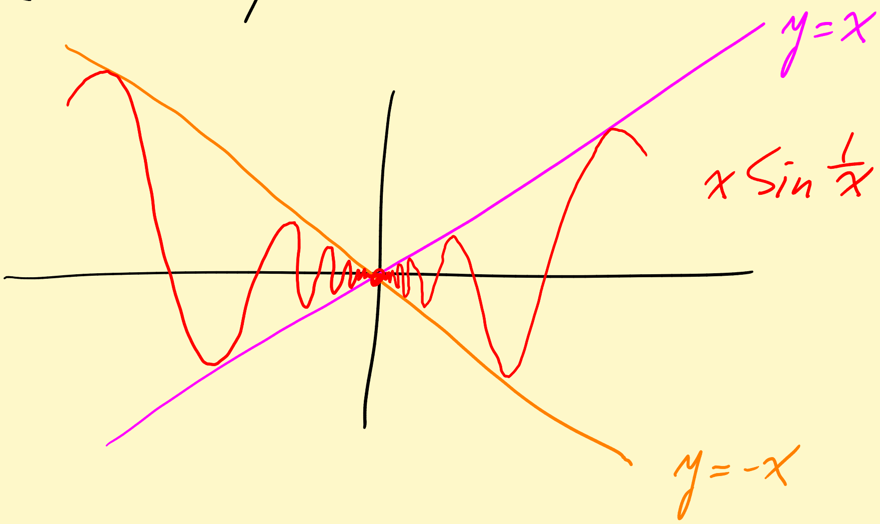
$x_n \to c$ as $n \to \infty$.

<u>Remark</u>: Useful when showing f is <u>not</u> continuous at c.

<u>EX</u>:
$$f(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$$

is <u>discontinuous</u> at every x.

EX



$$f(x) = \begin{cases} 0 & x = 0 \\ x\sin\frac{1}{x} & x \neq 0 \end{cases}$$ is continuous on $\mathbb{R}$.

$\sin\frac{1}{x}$ is not continuous at $x=0$. It is on $\mathbb{R} - \{0\}$.