Lecture 22 Combinations of continuous functions. (5.2)

Exam 1. Ave = 73

A ≥ 80
B ≥ 65
C ≥ 55

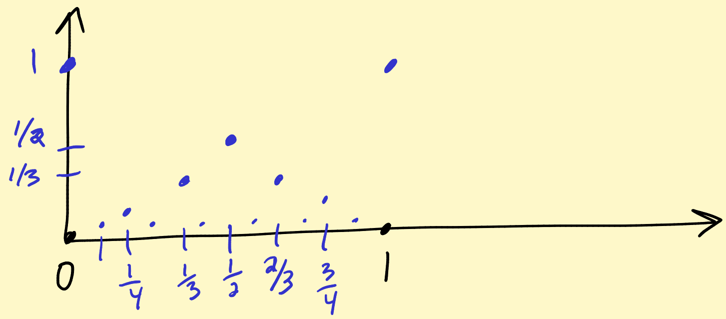
Thomae's fcn

$$T(x) = \begin{cases} 0 & x \in [0,1] - \mathbb{Q} \\ \frac{1}{q} & x \in \mathbb{Q},\ x = \frac{p}{q} \end{cases}$$

$p, q \in \mathbb{N}$, no common factors

$T(0) = 1$, $T(1) = 1$

T is continuous at each $x \in [0,1] - \mathbb{Q}$ (irrational).
discont at each $x \in \mathbb{Q}$ in $[0,1]$.



Easy facts If f and g are continuous on $A \subset \mathbb{R}$, then so are

1) $c_1 f + c_2 g$, c_1, c_2 constants.

2) fg

3) $|f|, |g|$

4) and, if $g(x_0) \neq 0$, $x_0 \in A$, then f/g is continuous at x_0.

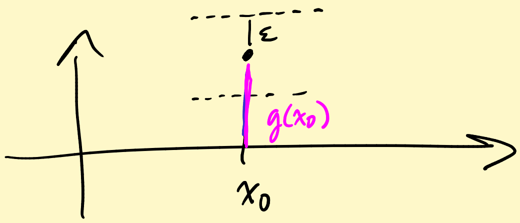
5) If $f(x) \geq 0$, then $\sqrt{f(x)}$ is continuous.

Pf's easy. For #4, need fact: If $g(x_0) \neq 0$ and

$\lim\limits_{x \to x_0} g(x) = g(x_0)$, then there is a $\delta > 0$ and $r > 0$ such that

$|g(x)| > r$ when $|x - x_0| < \delta$ and $x \in A$.

Why: Let $\varepsilon = \frac{g(x_0)}{2}$ and mess around with inequalities



$r = \frac{g(x_0)}{2}$ too.

Composition of continuous fcns If $A, B \subset \mathbb{R}$ and $g: A \to \mathbb{R}$, $f: B \to \mathbb{R}$ are continuous, and $g(A) \subset B$, then $f \circ g$ is continuous on A.

Why: Suppose $c \in A$. Then $g(c) \in B$, and f is cont. at $g(c)$. So, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(y) - f(g(c))| < \varepsilon \quad \text{when} \quad |y - g(c)| < \delta$$

$\delta = \varepsilon_2 > 0$ a new ε.

Know g cont at c. So there is a $\delta_2 > 0$ such that

$$|g(x) - g(c)| < \varepsilon_2 = \delta \quad \text{when} \quad |x - c| < \delta_2.$$

Let $y = g(x)$ and go backwards: $|f(g(x)) - f(g(c))| < \varepsilon$

when $|x-c|<\delta_2$ $(x \in A)$.

Consequences: Polynomials are cont on $\mathbb{R}$.

Book: Easy: constants are cont. ✓ x is continuous $(\varepsilon=\delta)$ ✓

$x^2 = x \cdot x$ is cont, $x^3 = x^2 \cdot x$, ...

linear combos are cont. ✓

Rational fcns $\frac{p(x)}{q(x)}$ are continuous at pts x where $q(x) \neq 0$.

$$\left[DQ = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \to 4 \text{ as } x \to 2. \right]$$

Book: Sin, Cos are Lip^1 (Lipschitz-1)

and consequently, they are continuous.

$$|\sin\alpha - \sin\beta| \le |\alpha-\beta|$$ ← $C=1$ in Lip^1 def^n

HWK: $Lip^1 \Rightarrow$ continuous.

Lip^α: $\quad |f(x)-f(y)| \le C\,|x-y|^\alpha \qquad (\alpha>0)$

5.3 Continuous fcns on a closed interval.

Big facts: Cont fcn on a closed interval is bounded, and it assumes its max and min at pts in the interval.

<u>Def^n</u> A fcn f is <u>bounded</u> on a set $A \subset \mathbb{R}$ means, there is an $M > 0$ such that $|f(x)| \le M$ for all $x \in A$.

<u>Thm</u>: Suppose f is continuous on $[a,b]$. Then f is <u>bounded</u> there.

<u>Why</u>: Suppose it isn't. Then there is an $x_1 \in [a,b]$ where $|f(x_1)| > 1$. There is an $x_2 \in [a,b]$ where $|f(x_2)| > 2$. Etc. Get (x_n) in $[a,b]$ such that $|f(x_n)| > n$.

Aha! (x_n) is a <u>bounded</u> seq (since $a \le x_n \le b$). Bolzano–Weierstraß yields a <u>convergent</u> subseq $x_{n_k} \to x_0$. $[a,b]$ contains all its cluster pts. ($[a,b]$ is closed.) Ouch! $f(x_{n_k}) \to f(x_0)$. $\lightning$

Can't because $|f(x_{n_k})| > n_k \ge k$.

<u>Remark</u>: $|f(x_{n_k})| > k$. Set of indicies n_k where

$f(x_k)$ is positive $\mathcal{I}^+$, or negative $\mathcal{I}^-$

can't both be finite. So at least one of them

is infinite. So can take a further subseq such

that either $f(x_{n_k}) > K$ $\quad (f(x_{n_k}) \to +\infty)$

or $f(x_{n_k}) < -K$ $\quad (f(x_{n_k}) \to -\infty)$

Similar ideas can be used to get max (or mins) attained.

Will show: $\text{Max} = \text{Sup}\{f(x) : x \in [a,b]\} = S$