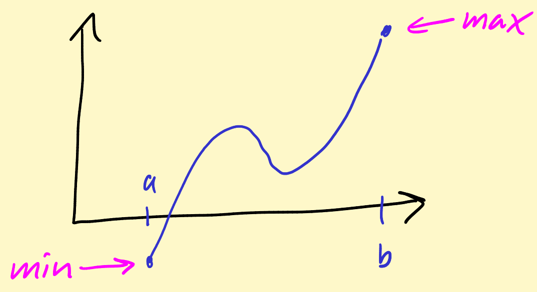
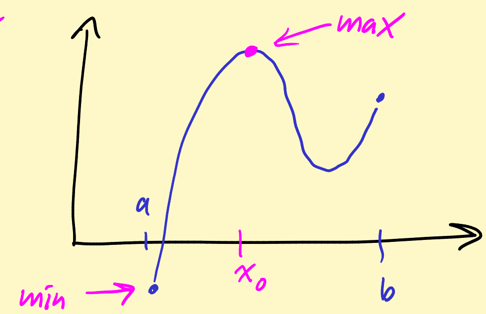
Lecture 23 The Intermediate Value theorem (5.3)

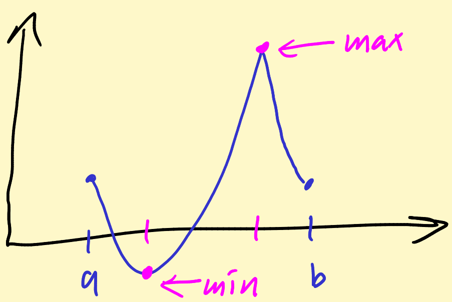
Last time: A continuous fcn on a closed interval is bounded.

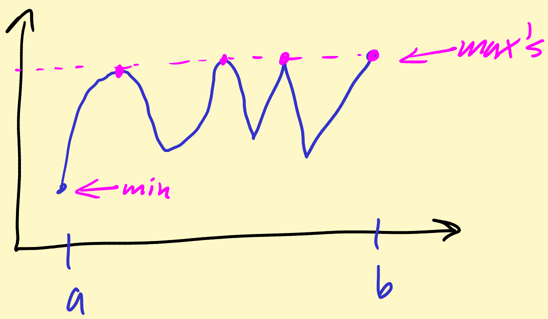
This time: A continuous fcn on a closed interval assumes its max and min values. [Absolute max and min – Book.]

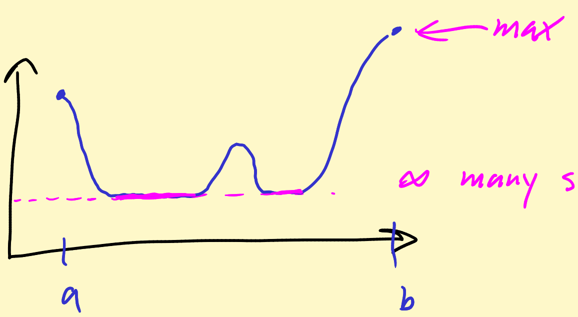
Examples





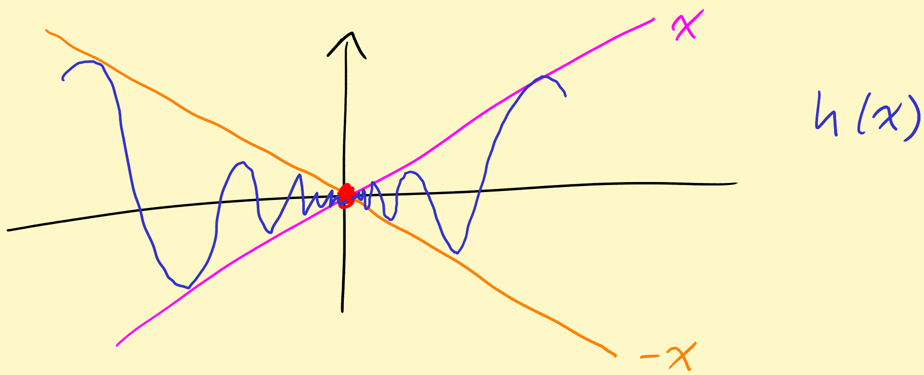


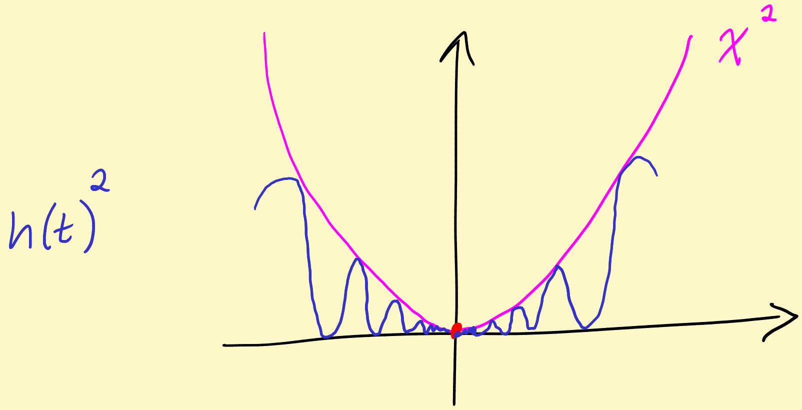




∞ many spots where min is attained

Fun example: $h(x) = \begin{cases} 0 & x = 0 \\ x \sin\frac{1}{x} & x \neq 0 \end{cases}$ continuous on $[-1,1]$.



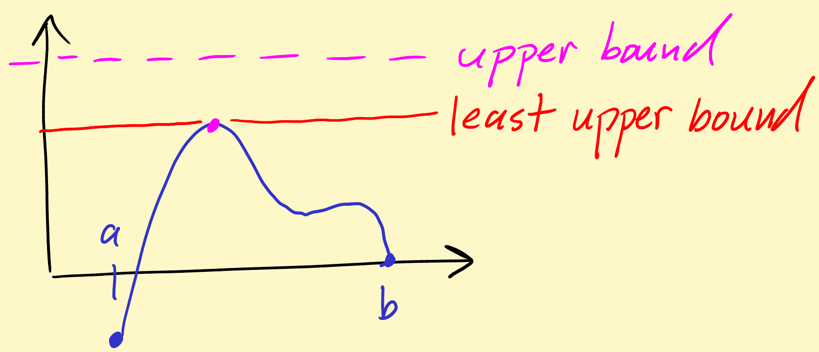


Aha! on $[-1,1]$, the min is attained at ∞ many isolated pts $\neq 0$ and at 0, which is a cluster pt of min pts!

$2-h(t)^2$ does same thing with max's.

Remark: No such thm for non-closed intervals (because max and min $=$ sup and inf can happen at endpoints).

Proof of theorem



Expect $M = \max = \sup\{f(x) : x \in I\} = \sup f(I) = S'$

Recall property: If $S' = \sup f(I)$ and $\varepsilon > 0$, then there must be a pt y in $f(I)$ such that $y > S' - \varepsilon$.

[If not, $S' - \varepsilon$ would be a smaller upper bound. $\lightning$]

Let $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$. $y_n \in f(I)$: so $y_n = f(x_n)$ for some $x_n \in I$.

$$S' - \frac{1}{n} < f(x_n) \leq S'$$

Bolzano-Weierstraß! $x_n \in I$. So (x_n) is a <u>bound</u> sequence. So there is convergent subseq $x_{n_k} \to c$.

I closed, so $c \in I$.

$$a \leq x_n \leq b$$
$$\downarrow$$
$$a \leq c \leq b$$

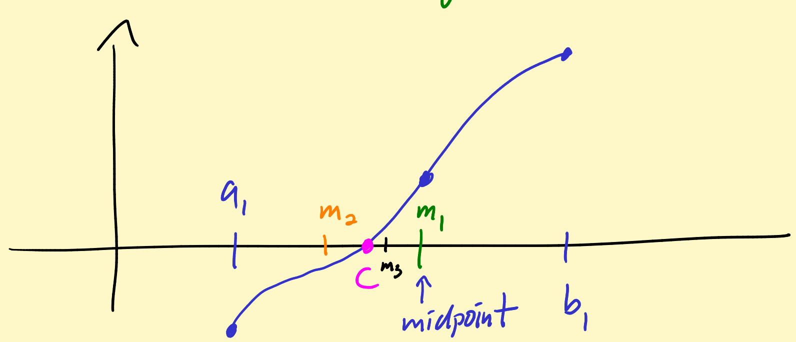
Squeeze thm

$$\begin{array}{ccccc} S' - \frac{1}{n_k} & < & f(x_{n_k}) & \leq & S' \\ \downarrow & & \downarrow \leftarrow f \text{ continuous} & & \downarrow \\ S' & \leq & f(c) & \leq & S' \end{array}$$

So $f(c) = S'$. We may call $S' = f(c)$ the <u>absolute</u> <u>max</u> M.

Similar reasing for min $m = \inf f(I)$.

<u>Bisection method for finding zeroes.</u>



$f(a) < 0$

$f(b) > 0$

There is a c with $a < c < b$ with $f(c) = 0$.

$m = \frac{a_1 + b_1}{2}$. If $f(m) > 0$, $a_2 = a_1$, $b_2 = m$. New interval.

If $f(m) < 0$, $a_2 = m$, $b_2 = b_1$. New interval

If $f(m) = 0$, QUIT! We found a zero!

Note $b_2 - a_2 = \frac{b_1 - a_1}{2}$.

Repeat! Get nested seq of closed intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots$$

$$b_n - a_n = \frac{1}{2^{n-1}}(b_1 - a_1) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

So $\bigcap_{n=1}^{\infty} I_n = \{p\}$, a single pt in $[a_1, b_1]$.

Claim $f(p) = 0$.

$$\underbrace{\begin{matrix} f(a_n) < 0, & f(b_n) > 0 \\ \downarrow & \downarrow \\ f(p) \le 0, & f(p) \ge 0 \end{matrix}}_{\text{must be that } f(p)=0}$$

Note: $a_n \le p \le b_n$ and $b_n - a_n \to 0$ implies $a_n \to p$ and $b_n \to p$.

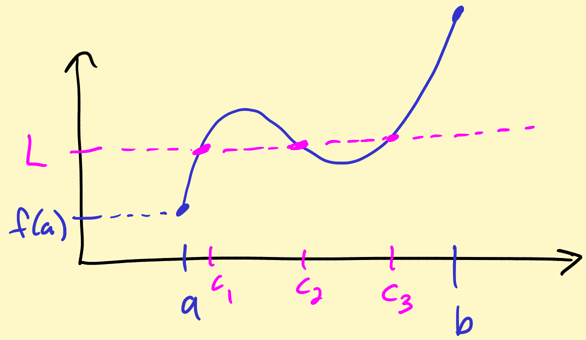
Can apply this routine to $f(x) - L$ to prove

Intermediate value theorem: If f is continuous on a closed interval $[a,b]$ and $f(a) < L < f(b)$

or $f(a) > L > f(b)$, there is a c with

$a < c < b$ where $f(c) = L$.

Pf: Bisection for $f(x) - L$ or $L - f(x)$.



Hmmm. Start with f on I. Get min m, max M.

Get $p_1 \in I$ where $f(p_1) = m$, $p_2 \in I$ where $f(p_2) = M$.

Case $p_1 = p_2$. $f = \text{const}$.

$p_1 < p_2$. IMT for $[p_1, p_2]$

$p_1 > p_2$. IMT for $[p_2, p_1]$.

See everything $m \le L \le M$ gets hit.

So $f(I) = [m, M]$.

Continuous fcns map closed intervals to closed intervals.

Think about $f((a,b))$, $f([a,b))$, $f((a,b])$ cases.
Image is an interval of some kind.