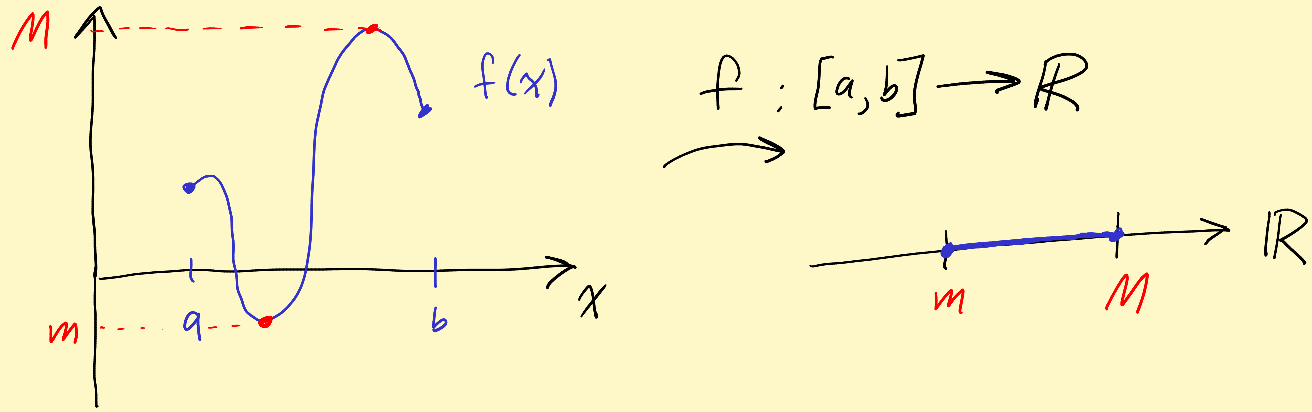
Lesson 24 More about continuous functions (5.3)

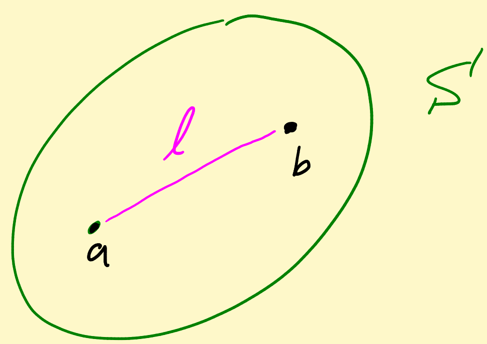


Fact: If f is continuous on a closed interval I, then

$$f(I) = [m, M] \quad \text{where} \quad \begin{array}{l} m = \min \text{ of } f \text{ on } I = \inf \\ M = \text{Max} \cdots \qquad = \sup \end{array}$$

Why: Two key facts: 1) Know min and Max are attained.
2) Intermediate Value Theorem

Convex sets in $\mathbb{R}^2$:



S is convex: If $a, b \in S$, then $l \subset S$.

"Convex set" in $\mathbb{R}$: Interval! (a,b), $[a,b)$, $(a,b]$, $[a,b]$
$(-\infty,\infty)$, $(-\infty,a)$, $(-\infty,a]$,
(b,∞), $[b,\infty)$

Fact: S is an interval if and only if, for every pair $x_1 \le x_2 \in S'$, $[x_1, x_2] \subset S'$.

Bounded intervals: $a = \inf S'$, $b = \sup S'$.

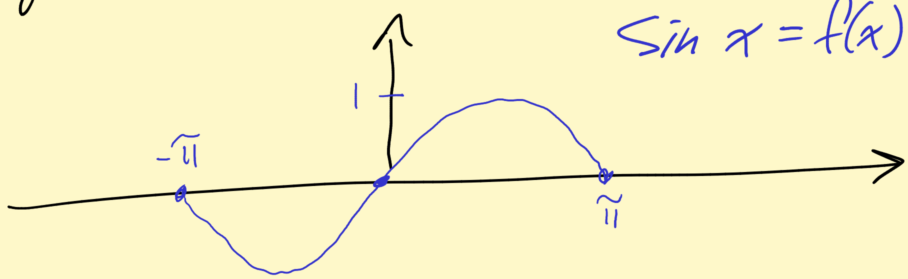
Preservation of intervals theorem: If f is continuous on an interval I, then $f(I)$ is a interval.

Remarks: Pf is just the intermediate value thm.

closed $\to$ closed is only straightforward case.

rest, get jumbled.

EX:



$$f\left(\left(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right)\right) = \left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right)$$

$$f\left((-\pi, \pi)\right) = [-1, 1]$$

$\sin(1/x)$ on $(0, \pi)$: $(0, \pi) \to [-1, 1]$

$\frac{1}{x}$ on $(0, \pi)$: $(0, \pi) \to \left(\frac{1}{\pi}, \infty\right)$

p. 133: 3. Find f, g discontinuous with

a) $f+g$ continuous, b) fg continuous.

Ideas: $\underbrace{(f+u)}_{F} + \underbrace{(g-u)}_{G} = f+g$ $\qquad$ f,g good, u bad, F,G bad

$$\frac{f}{g} = \frac{(fu) \leftarrow F}{(gu) \leftarrow G}$$

Hmmm: $f_1(x) = \begin{cases} 0 & \mathbb{Q} \\ 1 & \mathbb{R}-\mathbb{Q} \end{cases}$

$$f_2(x) = \begin{cases} 1 & \mathbb{Q} \\ 0 & \mathbb{R}-\mathbb{Q} \end{cases}$$

Find an example where f is discontinuous, but $|f|$ is continuous. Replace 0 by -1 above!

Summer reading. "A primer of real analysis," Ralph P. Boas (+Harold Boas).

"Most" continuous fcns are nowhere diff'ble. (Baire category theorem)

Cardinality of $[0,1]$ is same as that of $[0,1]\times[0,1]$.

$$\left[\exists\, \varphi : [0,1] \to [0,1]\times[0,1] \text{ one-to-one onto.}\right]$$

But, there is no <u>continuous</u> φ that does that.

<u>But</u>, there is a continuous "space filling"

curve $\mathcal{C}$ that is <u>onto</u>.

Google "Peano curves" or "space filling curves".

p. 134 ; 8. $f, g : [0,1] \to \mathbb{R}$. $f(r) = g(r)$ for $r \in [0,1] \cap \mathbb{Q}$.

Is $f \equiv g$ on $[0,1]$? No! $f = \begin{cases} 1 & [0,1] \cap \mathbb{Q} \\ 0 & [0,1] - \mathbb{Q} \end{cases}$

$g \equiv 1$.

But, if f, g continuous, then <u>yes</u>.

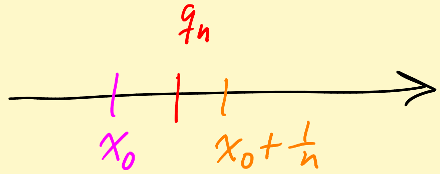
Pick $x_0 \in [0,1] - \mathbb{Q}$, irrational pt.

Sequential def^n of continuous at x_0:

f cont at x_0 $\Longleftrightarrow$ $f(x_n) \to f(x_0)$ for every seq in $[0,1]$ with $x_n \to x_0$ as $n \to \infty$.

Look at $f - g$. It's zero on $[0,1] \cap \mathbb{Q}$.

Rationals are <u>dense</u>.



Get (q_n) with $q_n \to x_0$.

$f - g$ is cont. $\lim\limits_{n \to \infty} \underbrace{\left(f(q_n) - g(q_n)\right)}_{0} = f(x_0) - g(x_0)$.