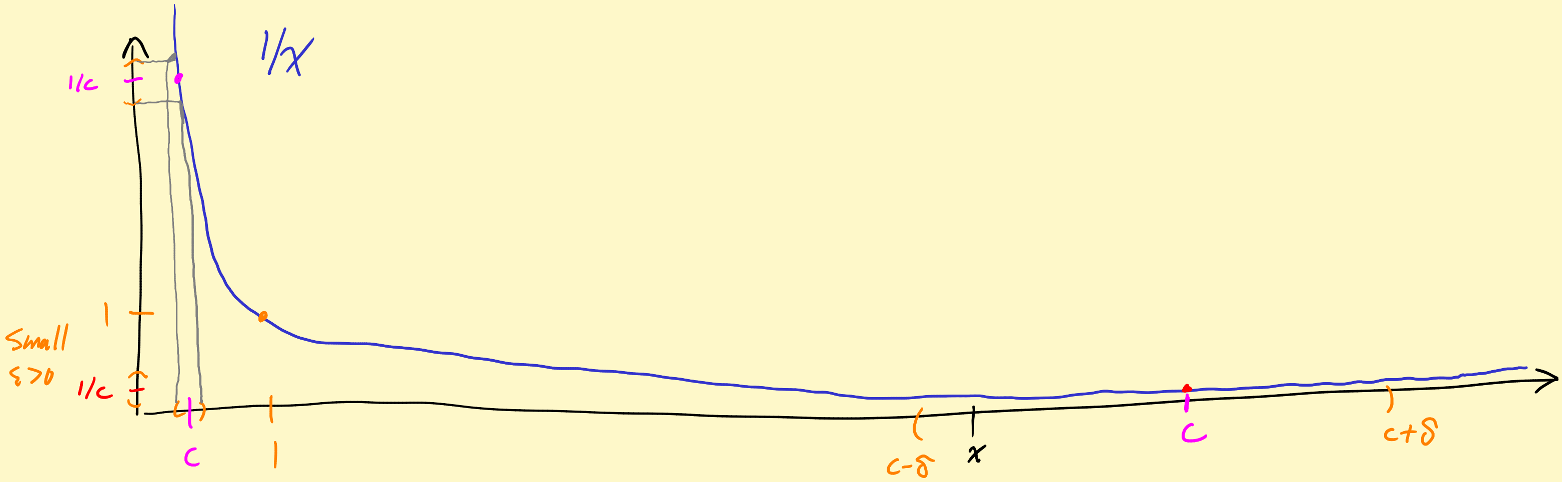
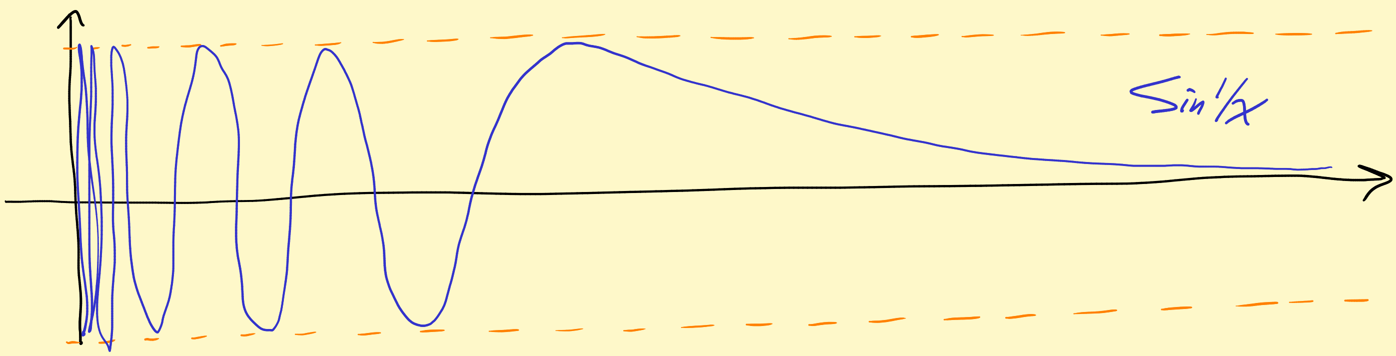
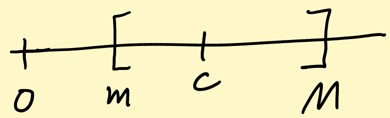
Lesson 25 Uniform continuity (5.4)





EX: $f(x)=\frac{1}{x}$. Given $\varepsilon>0$.



$$|f(x)-f(c)| = \left|\frac{1}{x}-\frac{1}{c}\right| = \left|\frac{c-x}{xc}\right|$$

$m<|x|$

$m<|c|$

$m^2<|xc|$

So $\frac{1}{|xc|}<\frac{1}{m^2}$

$$< \frac{1}{m^2}|c-x| \overset{\text{want}}{<} \varepsilon$$

Need $|c-x|<\underbrace{\varepsilon m^2}_{\delta=\varepsilon m^2}$

works for all c in $[m,M]$

and $x\in[m,M]$, $|x-c|<\delta$.

Def^n: $f:A\to\mathbb{R}$ is uniformly continuous if, given $\varepsilon>0$, there is a $\delta>0$ such that,

$|f(x) - f(c)| < \varepsilon$ when $x, c \in A$ and

$|x - c| < \delta$. [Same δ works for all $c \in A$.]

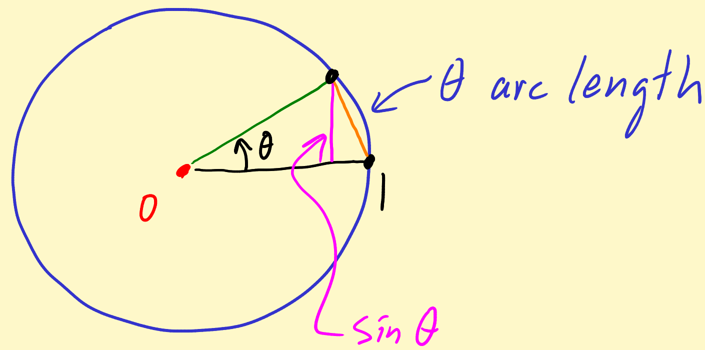
EX: $\sin x$ is uniformly continuous on $\mathbb{R}$.

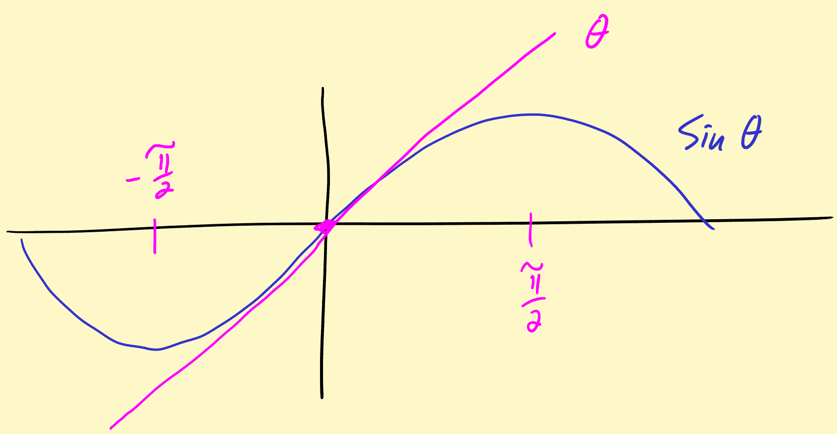
Basic facts: 1) $|\cos\theta| \le 1$, $|\sin\theta| \le 1$ all θ.

(easy leg $\le$ hyp.)

2) $|\sin\theta| \le |\theta|$

(θ radians)





(outside $(-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad |\sin\theta| \le 1 < |\theta|$ ✓)

Trig identities: $\sin(\alpha+\beta) = \cos\alpha \sin\beta + \cos\beta \sin\alpha$

$$\sin x - \sin c = 2 \sin\frac{x-c}{2} \cos\frac{x+c}{2}$$

Nice trick. Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

$$\underbrace{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}_{\left(\cos\alpha\cos\beta - \sin\alpha\sin\beta\right) + i\left(\cos\alpha\sin\beta + \cos\beta\sin\alpha\right)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

equal!

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha$$

$\cos(-\beta) = \cos\beta$

$$\sin(\alpha-\beta) = \cos\alpha(-\sin\beta) + \cos\beta\sin\alpha$$

$$\sin\underbrace{(\alpha+\beta)}_{x} - \sin\underbrace{(\alpha-\beta)}_{c} = 2\cos\underbrace{\alpha}_{\frac{x+c}{2}}\sin\underbrace{\beta}_{\frac{x-c}{2}}$$

$$\begin{cases} \alpha+\beta = x \\ \alpha-\beta = c \end{cases} \Leftarrow \beta = x - \alpha = x - \left(\frac{x+c}{2}\right) = \frac{x-c}{2}$$

$$2\alpha = x + c$$

$$\alpha = \frac{x+c}{2}$$

Aha! $$\left|\sin x - \sin c\right| = \left|2\cos\frac{x+c}{2}\sin\frac{x-c}{2}\right|$$

$$\le 2\underbrace{\left|\cos\frac{x+c}{2}\right|}_{\le 1}\underbrace{\left|\sin\frac{x-c}{2}\right|}_{\le \left|\frac{x-c}{2}\right|}$$

$$\left|\sin x - \sin c\right| \le |x - c|$$

Can use $\delta = \varepsilon$ for <u>all</u> c. [Uniform in c.]

Lipschitz-1 : $|f(x) - f(c)| \le K\,|x - c|$.

<u>Fact</u>: Lip-1 fcns are <u>uniformly</u> continuous.

<u>Big fact</u> A continuous fcn on a <u>closed</u> interval must be <u>uniformly</u> continuous there.

<u>Why</u>: Suppose $f : [a,b] \to \mathbb{R}$ is continuous, but <u>NOT</u> uniformly cont. Then there is $\varepsilon_0 > 0$ for which def^n fails. Then no $\delta > 0$ can be found such that $|f(x) - f(c)| < \varepsilon_0$ when $|x - c| < \delta$, $x, c \in [a,b]$.

Hence, for $\delta = \frac{1}{n}$, there are $x_n, c_n \in [a,b]$ with

$$|x_n - c_n| < \frac{1}{n}, \quad \text{but} \quad |f(x_n) - f(c_n)| \ge \varepsilon_0$$

Bolzano-Weierstraß thm! (x_n) and (c_n) are sequences in a closed interval, so they are bounded.

So (x_n) has a convergent subseq (x_{n_k}).

Then $x_{n_k} \to x_0$. Note: $x_0 \in [a,b]$ because $[a,b]$ is closed.

Claim: $c_{n_k} \to x_0$ too.

$$|c_{n_k} - x_0| = |c_{n_k} - x_{n_k} + x_{n_k} - x_0|$$

$$\leq \underbrace{|c_{n_k} - x_{n_k}|}_{< \frac{1}{n_k} \leq \frac{1}{k} \to 0} + \underbrace{|x_{n_k} - x_0|}_{\to 0}$$

as $k \to \infty$.

Know f is cont at x_0.

$$\left.\begin{array}{l} x_{n_k} \to x_0 \\ c_{n_k} \to x_0 \end{array}\right\} \quad \begin{array}{l} f(x_{n_k}) \to f(x_0) \\ f(c_{n_k}) \to f(x_0) \end{array}$$

Oops! But $|f(x_{n_k}) - f(c_{n_k})| \geq \varepsilon_0$ ↯

Our assumption that f is <u>NOT</u> unif cont is <u>NOT</u> true. So f <u>is</u> unif cont.

<u>Another important fact</u> If $f: (a,b) \to \mathbb{R}$ is <u>uniformly</u> continuous on (a,b), then f extends to $[a,b]$ as a <u>continuous</u> fcn on $[a,b]$ (and hence is unif cont on $[a,b]$) [meaning

$f(a)=A$ and $f(b)=B$ can be defined to make f cont on $[a,b]$.

<u>Key fact</u>: If f is <u>uniformly</u> continuous and (x_n) is a Cauchy seq, then $(f(x_n))$ is a Cauchy seq.