# Lesson 26 Monotone functions (5.6) skip 5.5

Continuous extension theorem If $f$ is uniformly continuous on $(a,b)$, then $f$ extends to $[a,b]$ in such a way that $f$ is continuous on $[a,b]$ (and so uniformly continuous there).

Why: Let $(x_n)$ be a seq in $(a,b)$ such that $x_n \to a$ as $n \to \infty$. Convergent sequences are Cauchy. Will show $(f(x_n))$ is also Cauchy. Let $\varepsilon > 0$. $f$ unif cont.

So $\exists$ $\delta$ such that $|x_n - x_m| < \delta$

$\Rightarrow$ $|f(x_n) - f(x_m)| < \varepsilon$. $(x_n)$ Cauchy $\Rightarrow$ there an $N$ such that $|x_n - x_m| < \delta$ when $n,m > N$.

Get $|f(x_n) - f(x_m)| < \varepsilon$ when $n,m > N$. ✓

$(f(x_n))$ is Cauchy. So $\lim\limits_{n\to\infty} f(x_n) = A$ exists.

Next, need to show that for every seq $(u_n)$ with $u_n \in (a,b)$ and $u_n \to a$, that $\lim\limits_{n\to\infty} f(u_n) = A$.

Easy. We know $\lim\limits_{n\to\infty} f(u_n) = B$ exists.

$$|A-B| = |A - f(x_n) + f(x_n) - f(u_n) + f(u_n) - B|$$

$$\leq \underbrace{|A - f(x_n)|}_{\to 0} + \underbrace{|f(x_n) - f(u_n)|}_{} + \underbrace{|f(u_n) - B|}_{\to 0}$$

$x_n \to a$

$u_n \to a$

So $x_n - u_n \to 0$ as $n \to \infty$.

unif cont. $\begin{cases} \text{Given } \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that} \\ |f(x_n) - f(u_n)| < \varepsilon \text{ when } |x_n - u_n| < \delta. \end{cases}$

$x_n - u_n \to 0$ as $n \to \infty$, So there is a $N$ such that $|x_n - u_n| < \delta$ when $n > N$.

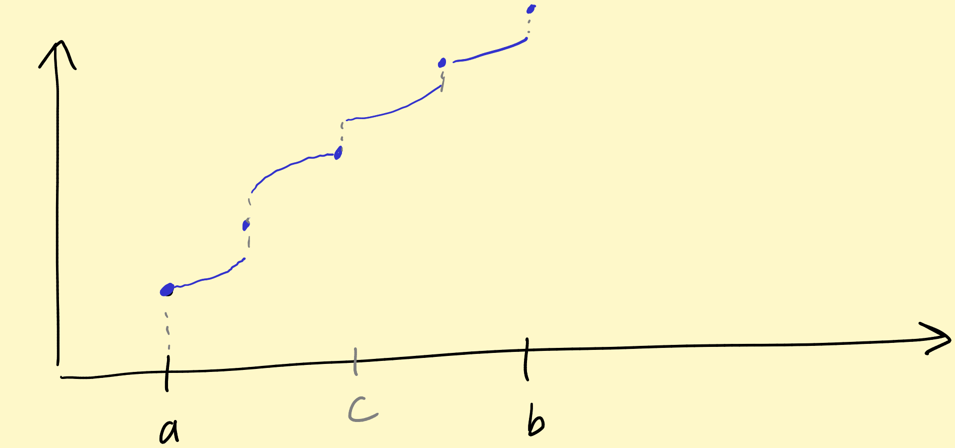
Middle term $\to 0$ too. Conclude $|A-B| = 0$.

So $A = B$. We've shown that $f(u_n) \to A$ for all seq in $(a,b)$ that tend to $a$. So $f$ is continuous up to $a$ if we define $f(a) = A$. Similar for $b$.

## Monotone fcns and continuity 5.6

Only do increasing case $[x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)]$.

Decreasing case similar.




<u>Easy facts</u>

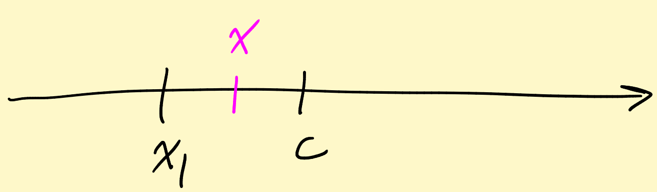
$$\lim_{x\to c^-} f(x) = \sup\{f(x) : x<c\}$$

$$\lim_{x\to c^+} f(x) = \inf\{f(x) : x>c\}$$

Note: Everything in inf set is $\geq$ everything in sup set. So $\lim\limits_{x\to c^-} \leq \lim\limits_{x\to c^+}$.

<u>Why</u> $\lim\limits_{x\to c^-} f(x) = \sup\limits_{x<c} f(x) = S'$

Let $\varepsilon>0$. Know there is an $x_1$ with $x_1<c$ such that $f(x_1) > S'-\varepsilon$.




Aha! If $x_1<x<c$, then $f(x_1)<f(x)\leq \underset{\sup}{\overset{\parallel}{S'}}$

Pick $\delta = c-x_1$. Then $|f(x)-S'|<\varepsilon$. ✓

Other side ($x > c$) similar.

Fact: $f$ continuous at $c \iff \lim_{x \to c^-} = \lim_{x \to c^+} = f(c)$.

Consequence: Increasing fcn $f$ is continuous at $c$

$$\iff \quad \sup_{x<c} f(x) = \inf_{x>c} f(x)$$

Def$^n$: $c$ is a "jump point" if

$$\inf_{x>c} f(x) > \sup_{x<c} f(x)$$

$$J(c) = \inf - \sup > 0.$$

( $f$ continuous where $J = 0$. )

Coolest thing since $\mathrm{card}(\mathbb{R}) > \mathrm{card}(\mathbb{N})$.

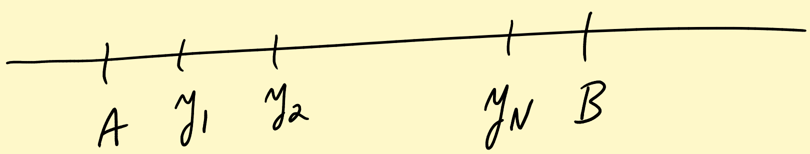
Thm: An increasing fcn on $[a,b]$ can have at most countably many jumps.

Why: Say $a \le x_1 < x_2 < \cdots < x_N \le b$ are finitely many jump pts in $[a,b]$.

(There might be more!)

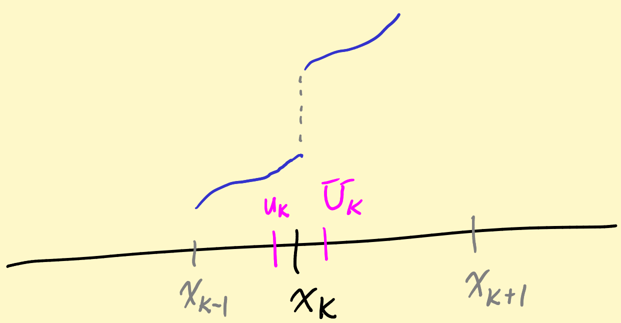
Then $\sum_{k=1}^{N} J(x_k) \le f(b) - f(a)$.

Book (why?)!




$$B - A = (B - y_N) + (y_N - y_{N-1}) + \cdots + (y_2 - y_1) + (y_1 - A)$$

Jump:




Aha! $f(\bar{v}_k) - f(u_k) \ge J(x_k)$

So $\sum_{k=1}^{N} J(x_k) \le \sum_{k=1}^{N} \big( f(\bar{v}_k) - f(u_k) \big) \le f(B) - f(A)$

How to use this to show only countably many jumps? Next time.