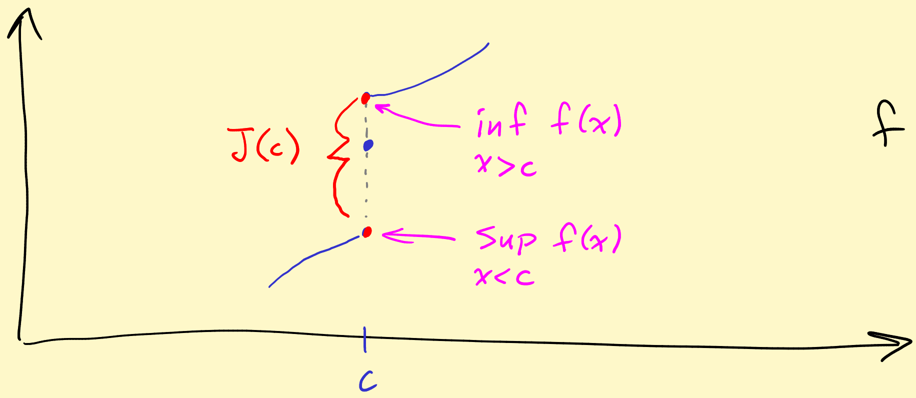
# Lesson 27 This and that before starting Differentiation (6.1)




f increasing

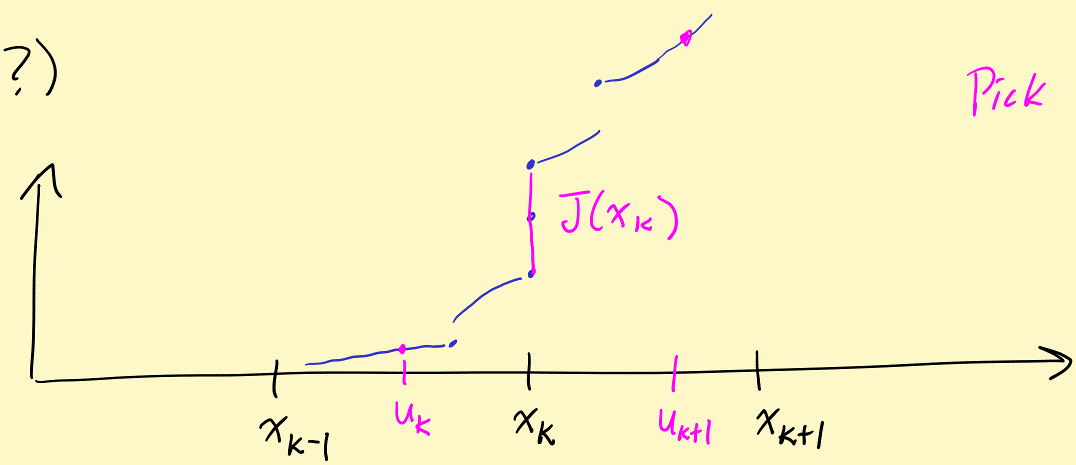
f is continuous at c $\iff$ $J(c) = 0$

**Thm**: A monotone fcn on $[a, b]$ can have at most countably many jumps.

Pf: Assume $a < x_1 < x_2 < \cdots < x_N < b$ are jump pts. Claim: $\sum_{n=1}^{N} J(x_n) \leq f(b) - f(a)$

(why?)

Pick $u_k$'s




Everything left of jump $\leq$ Everything right of jump

So $$f(u_k) \leq \sup_{x<c} f(x) \leq \inf_{x>c} f(x) \leq f(u_{k+1})$$

$$J(x_k) = \inf_{x>c} - \sup_{x<c} \leq f(u_{k+1}) - f(u_k)$$

$$\sum_{k=1}^{N} J(x_k) \leq \sum_{k=1}^{N} \left[ f(u_{k+1}) - f(u_k) \right] = \underset{\substack{\uparrow \\ x_N < u_{N+1} < b}}{f(u_{N+1})} - \underset{\substack{\uparrow \\ a < u_1 < x_1}}{f(u_1)}$$

$$f(a) \leq f(u_1) \leq f(u_{N+1}) \leq f(b)$$

✓ $$\sum J \leq f(u_{N+1}) - f(u_1) \leq f(b) - f(a)$$

How to use this:

How many jumps can be $> \frac{1}{n}$? $n \in \mathbb{N}$.

Say $a < x_1 < \cdots < x_N < b$ have $J(x_k) > \frac{1}{n}$

Then $$\sum_{k=1}^{N} J_k(x_k) > N \cdot \frac{1}{n}$$

Get $$\frac{1}{n} N < f(b) - f(a)$$

$$N < n\left[ f(b) - f(a) \right]$$

(Archimides princ.) Aha! # pts with jump $> \frac{1}{n}$ must be <u>finite</u>.

$$S' = \{x : J(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : J(x) > \tfrac{1}{n}\}$$

Easy fact A countable union of finite sets is countable!

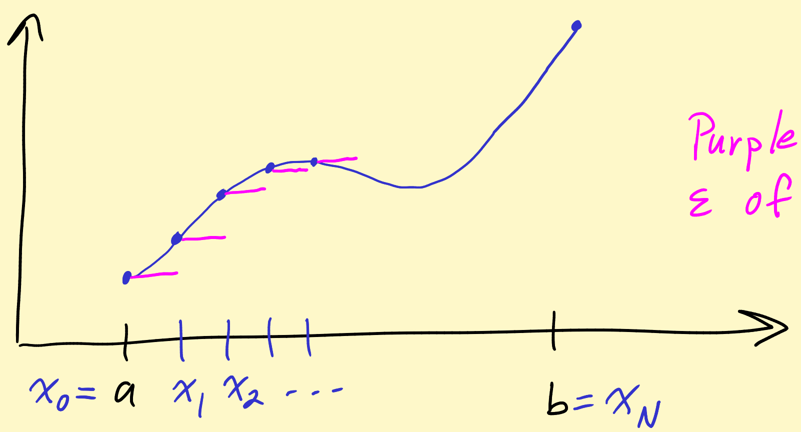
Before we move on to Chap 6 :

Fun thing: Can approximate continuous fcns on $[a,b]$ by step fcns.

Let $\varepsilon > 0$. $f$ continuous on $[a,b]$. Know that $f$ is uniformly cont on $[a,b]$. So there is a $\delta > 0$ such that $|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$.

Aha! Cut up $[a,b]$ into segments of size $< \delta$!

Need $\Delta x = \frac{b-a}{N} < \delta$. Pick $N \in \mathbb{N}$ that does this.




Purple fcn is within $\varepsilon$ of graph of $f(x)$ ✓

$x_n = a + n\Delta x$

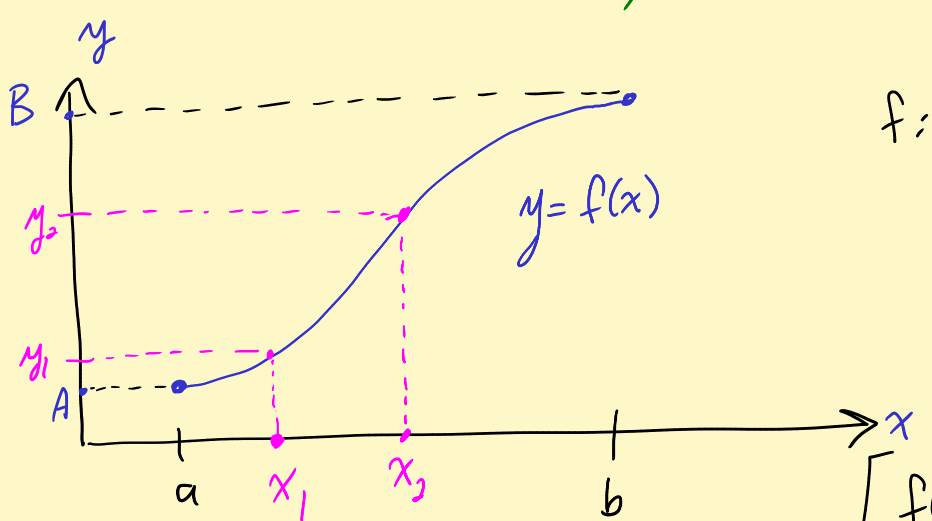
## Famous theorem (Weierstraß approximation theorem)

Given continuous fcn $f(x)$ on $[a,b]$ and $\varepsilon > 0$, there is a polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [a,b]$.

[Can uniformly approximate cont fcns on a closed interval by polynomials. Polynomials are "dense" among continuous fcns on a closed interval.]

## Continuous inverse function theorem for monotone fcns.

Increasing version: Suppose $f: [a,b] \to \mathbb{R}$ is a strictly increasing continuous fcn on $[a,b]$. Know $f$ maps $[a,b]$ to interval $[A,B]$. (Intermediate value thm means whole interval hit.)




$x = g(y)$

$f: [a,b] \to [A,B]$ onto, by IVT.

1-1, because it is strictly increasing.

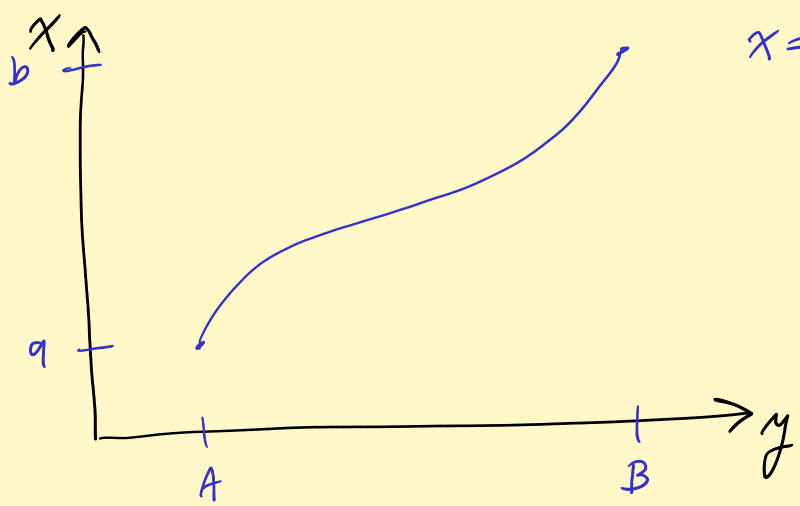
$$\left[\begin{array}{l} f(x_1) = y \\ f(x_2) = y \end{array}\right.$$

If $x_1 \neq x_2$, choose so that $x_1 < x_2$. Oops! Strictly increasing $\Rightarrow f(x_1) < f(x_2)$. $\lightning$ ]

Inverse fcn $g: [A,B] \to [a,b]$ exists.

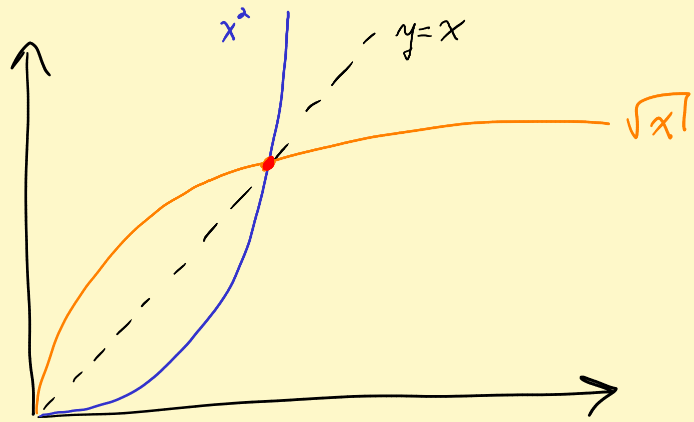
Easy to see: 1) $g$ is also strictly increasing.

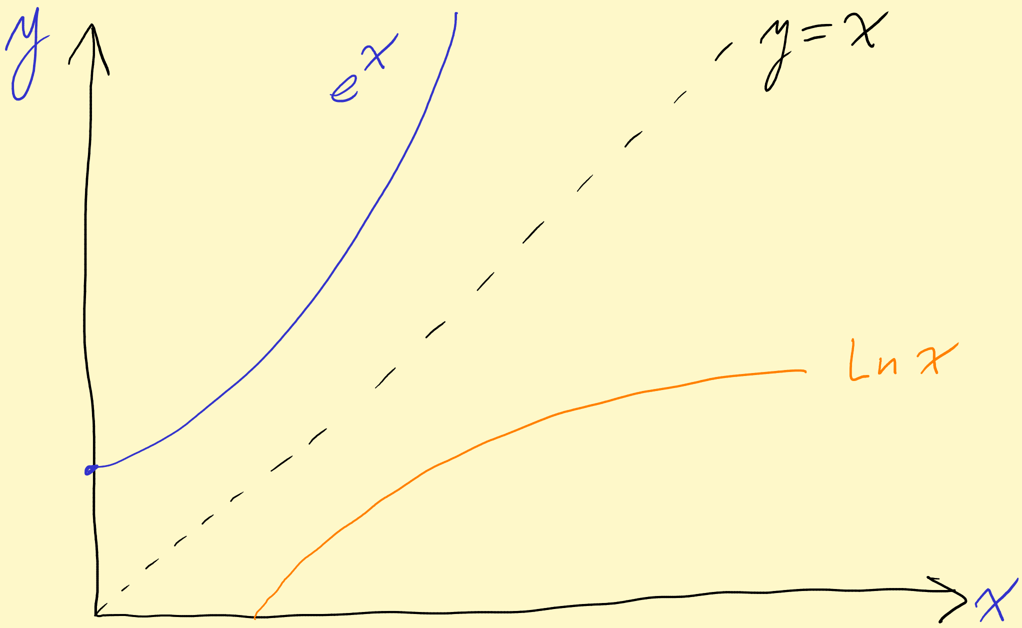
2) $g$ cannot have jumps. [Because pts in jumps get missed, and $g$, being 1-1, onto can't miss anything.]




$x = g(y)$ can't have jumps! So it is continuous

Continuous strictly monotone fcns have continuous strictly monotone inverses.






graph of $f^{-1}$ = reflection about $y=x$ of graph of $f$.