Lesson 28 Differentiation (6.1)

Highlights from 6.1 (Read all of 6.1 carefully.)

p. 141: 17 $f:[0,1]\to\mathbb{R}$ takes only rational values.

($f(x)\in\mathbb{Q}$ for every $x\in[0,1]$). Also, f continuous.

Then f must be constant.

Hints: IMT and density of $I-\mathbb{Q}$ in I.

Def^n: Suppose $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$.

If $\lim\limits_{x\to c}\underbrace{\dfrac{f(x)-f(c)}{x-c}}_{DQ}$ exists and is equal to L, we say f diff'ble at c and write

$f'(c)=L$.

Fact f diff'ble at c $\Rightarrow$ f continuous at c.

Why (*) $\dfrac{f(x)-f(c)}{x-c}-L=E(x)$ ← def^n of $E(x)$. "error at x" $x\neq c$.

$f'(c)$ exists: $E(x)\to 0$ as $x\to c$.

Define $E(c)=0$. Then $E(x)$ is continuous at c.

Solve (*) for $f(x)$: $f(x)=f(c)+L\cdot(x-c)+E(x)(x-c)$

$$\begin{array}{ccccccccc} f(x) = & f(c) & + & L \cdot & (x-c) & + & E(x) & (x-c) & \\ & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \text{as } x \to c \\ & f(c) & + & L \cdot & 0 & + & 0 \cdot & 0 & \end{array}$$

So $f(x) \to f(c)$ as $x \to c$. $\big(L = f'(c) \big)$

Nice formula: $f(x) = f(c) + f'(c)(x-c) + \underbrace{E(x)(x-c)}_{R(x)}$

where $E(x)$ is continuous at $x=c$ if we set $E(c)=0$. Remainder $R(x) \to 0$ fast as $x \to c$ since both $E(x)$ and $x-c \to 0$ as $x \to c$.

Favorite facts: $(fg)' = f'g + fg'$

$$DQ = \frac{f(x)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c}$$

$$= f(x)\frac{g(x)-g(c)}{x-c} + g(c)\frac{f(x)-f(c)}{x-c}$$

$f'(c)$ exists so $f(x) \to f(c)$

$x \to c$

$$= f(c) \cdot g'(c) + g(c) \cdot f'(c) \checkmark$$

Remark: $f(x) = x \qquad DQ = \frac{x-c}{x-c} = 1 \xrightarrow{!} 1.$

So $\underbrace{x \cdot x}_{x^2}$ is diff'ble

$x \cdot x^2$ is too ...

See polys are diff'ble. $\frac{d}{dx}(x^n) = n x^{n-1}$ etc.

2) $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ where $g(x) \neq 0$.

why Check $\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$ where g non-vanishing.

$$DQ = \frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x-c} = \frac{\left[-\frac{g(x)-g(c)}{g(x)g(c)}\right]}{x-c}$$

$$= -\left[\frac{g(x)-g(c)}{x-c}\right] \cdot \left(\frac{1}{g(x)g(c)}\right)$$

[Need to know $g(c) \neq 0$, g continuous at c implies $\Rightarrow$ there is $m > 0$ and a $\delta > 0$ such that $|g(x)| > m$ when $|x-c| < \delta$.]

$$= -\left[\frac{g(x)-g(c)}{x-c}\right]\cdot\left(\frac{1}{g(x)g(c)}\right)$$

$$\downarrow$$

$$-\ g'(c)\cdot\frac{1}{g(c)}\cdot\frac{1}{g(c)} \quad \checkmark$$

Next $\left(\frac{f}{g}\right)' = \left[f\cdot\frac{1}{g}\right]' = f'\cdot\frac{1}{g} + f\left(\frac{-g'}{g^2}\right)$

$=$ quotient formula.

<u>Chain rule</u>: $h(x) = f(g(x))$

then $h'(x) = f'(g(x))\,g'(x)$

Freshman calc: $\dfrac{f(g(x)) - f(g(c))}{x-c} =$

$$\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\cdot\frac{g(x)-g(c)}{x-c}$$

$g(x)\to g(c)$ as $x\to c$.

$\dfrac{f(y)-f(C_1)}{y-C_1}$ where $y = g(x)$, $C_1 = g(c)$

Hmmm. What if $g(x)-g(c)=0$ for x close to c???

Easy and correct proof:

Know $f(y)$ is diff'ble at $y=g(c)$.

So (*) $\boxed{f(y)=f(g(c))+f'(g(c))\,(y-g(c))+E(y)\cdot(y-g(c))}$

where $E(y)\to 0$ as $y\to g(c)$. [Let $E(g(c))=0$].

Now let $y=g(x)$ in (*) and mess around:

$$DQ=\frac{f(g(x))-f(g(c))}{x-c}=f'(g(c))\,\frac{[g(x)-g(c)]}{x-c}+E(g(x))\,\frac{(g(x)-g(c))}{x-c}$$

$$\xrightarrow{x\to c}\quad f'(g(c))\cdot g'(c)+0\cdot g'(c)$$

Need $E(g(c))=0$ ✓, E cont at $g(c)$ ✓,

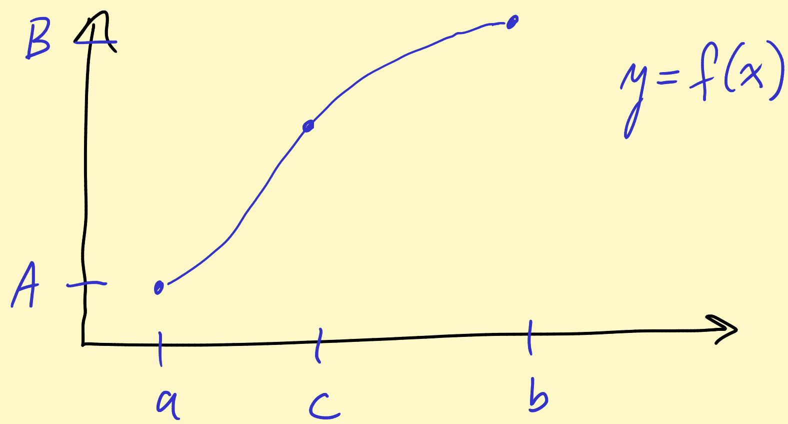
$g(x)\to g(c)$ as $x\to c$ ✓ [because $g'(c)$ exists $\Rightarrow g$ cont.]

Inverse function theorem (Diff'ble version)

Continuous version: f cont and strictly monotone on $[a,b]$, then f^{-1} exists and is also continuous and strictly montone. (last time).

Diff'ble version: keep Hyp of continuous version and add $f'(c) \neq 0$.

Increasing case.



$f'(c) > 0$.

$$\boxed{(f^{-1})'(f(c)) = \frac{1}{f'(c)}}$$

Hard part of Inverse fcn thm: Showing that f^{-1} is continuous. We did that last time!

Let $g(y) = f^{-1}(y)$. Think $y = f(x)$.

$$DQ = \frac{g(y) - g(f(c))}{y - f(c)} = \frac{1}{\left[\dfrac{y - f(c)}{g(y) - g(f(c))}\right]}$$

$$= \frac{1}{\left[\dfrac{f(x) - f(c)}{x - c}\right]}$$

Aha! g cont., $f(x) \to f(c)$ as $x \to c$.

$$x = g(f(x)) \to c \quad \text{as } x \to c.$$