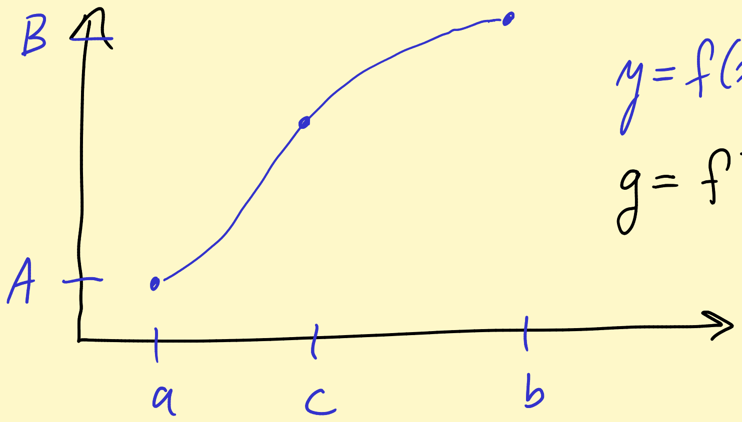
Lesson 29 Mean value theorem (6.2)

Diff'ble inverse function theorem (from last time)

Increasing case.



$y = f(x)$

$g = f^{-1}$

assumption

$\boxed{f'(c) > 0.}$

$(f^{-1})'$ exists at $f(c)$ &

$$\boxed{(f^{-1})'(f(c)) = \frac{1}{f'(c)}}$$

Knowing that $g = f^{-1}$ is continuous makes proof a snap:

$$DQ = \frac{g(y) - g(f(c))}{y - f(c)} = \frac{1}{\left[\dfrac{y - f(c)}{g(y) - g(f(c))}\right]}$$

$$\boxed{\begin{array}{l} y = f(x) \\ x = g(y) \end{array}}$$

$$= \frac{1}{\left[\dfrac{f(x) - f(c)}{x - c}\right]}$$

$$g'(f(c)) = \lim_{y \to f(c)} DQ$$

Know: $g(y) \to g(f(c))$ as $y \to f(c)$ [g^{-1} is cont.]

So $x = g(y) \longrightarrow g(f(c)) = c$ as $y \to f(c)$.

So $DQ \longrightarrow \frac{1}{f'(c)}$ from formula above.

How to remember $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$

$g = f^{-1}$. So $g(f(x)) = x$

Chain rule! $\quad g'(f(x))f'(x) = \frac{d}{dx}(x) = 1$

$$g'(\underbrace{f(x)}_{y}) = \frac{1}{f'(x)} \checkmark$$

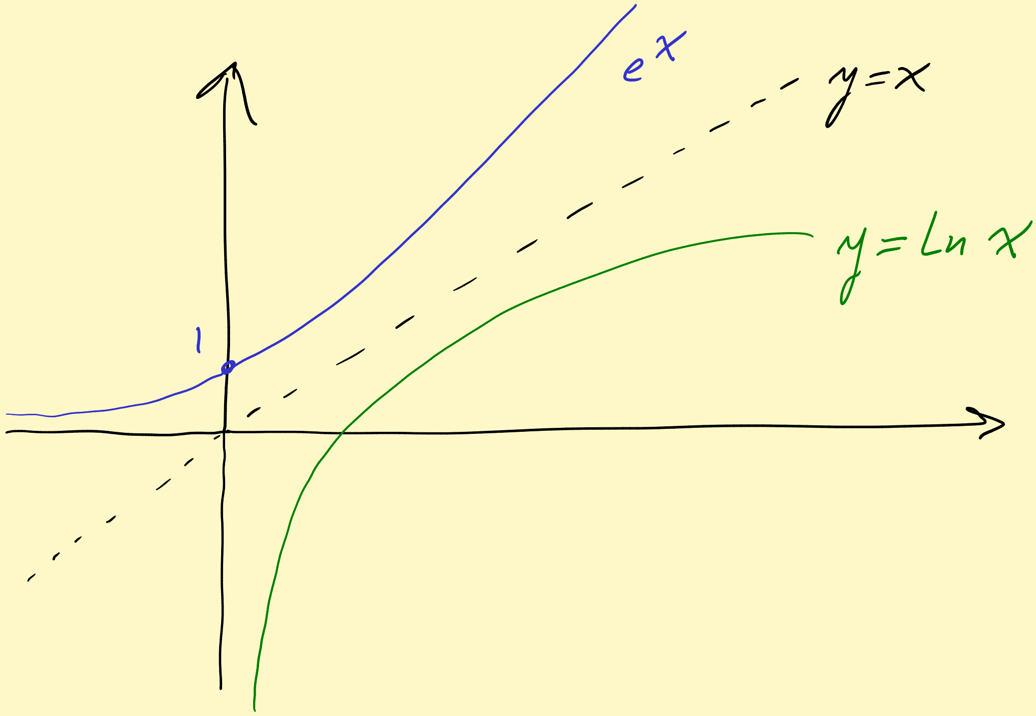
$x = g(y)$

$$g'(y) = \frac{1}{f'(g(y))}$$

EX: e^x is strictly increasing, diff'ble ($\Rightarrow$ cont).

$\frac{d}{dx}(e^x) = e^x > 0$. So the log fcn

$\ln x$ exists and is diff'ble.

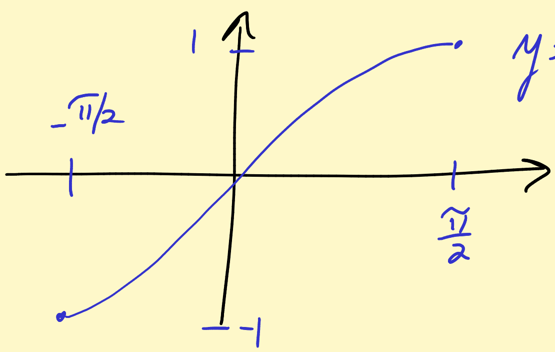


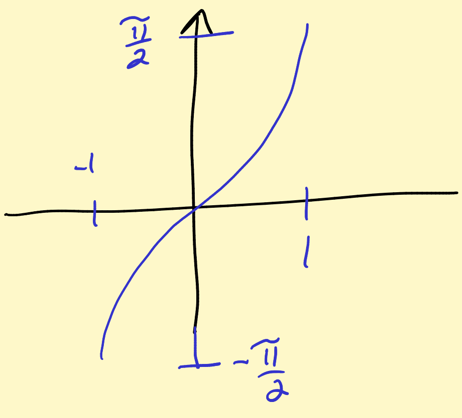
$$e^{\ln x} = x$$

Chain rule $\quad e^{\ln x} \cdot \frac{d}{dx} \ln x = 1$

$$\frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x} \quad \checkmark$$

EX:



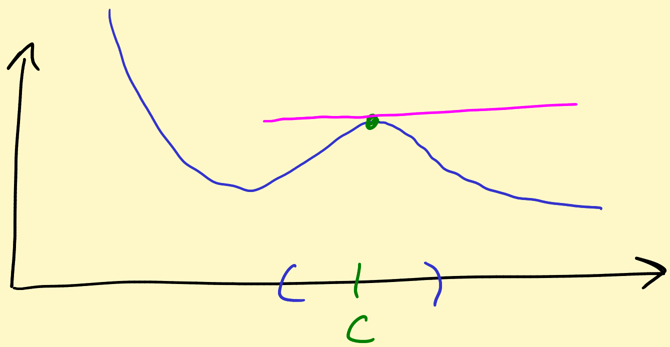


$$\frac{d}{dx}\operatorname{ArcSin} x = \frac{1}{\sqrt{1-x^2}}$$

Pf: Chain rule, $\frac{d}{dx}\sin x = \cos x$, $\sin^2 x + \cos^2 x = 1$

Easy fact If f has a local max at c (book: relative max) and $f'(c)$ exists, then $f'(c)=0$.

Same true for mins.



DQ on left $\quad \dfrac{f(x)-f(c)}{x-c} \begin{matrix} \leftarrow \le 0 \\ \leftarrow < 0 \end{matrix} \Bigg\} \ge 0$

DQ on right $\quad \dfrac{f(x)-f(c)}{x-c} \begin{matrix} \leftarrow \le 0 \\ \leftarrow > 0 \end{matrix} \Bigg\} \le 0$

$$\lim_{x\to c^-} DQ \ge 0 = \lim_{x \to c+} DQ \le 0$$

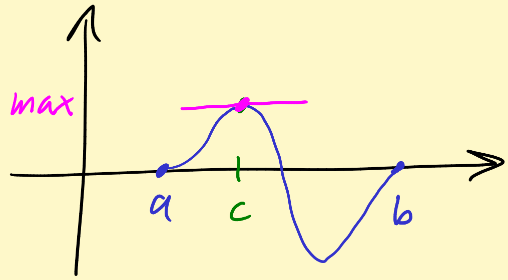
must be $=$. Both must be zero!

Rolle's thm: Suppose f continuous on $[a,b]$ and diff'ble on (a,b) and $f(a)=f(b)=0$. Then there is a c with $a < c < b$ where $f'(c) = 0$.

Pf: Know f assumes its max and min on $[a,b]$.

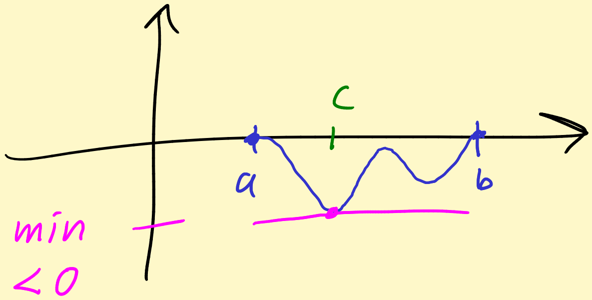
Dumb case. $\max = \min = 0$. Then $f \equiv 0$. All $a < c < b$ work.

Case $\max > 0$



$f'(c) = 0$

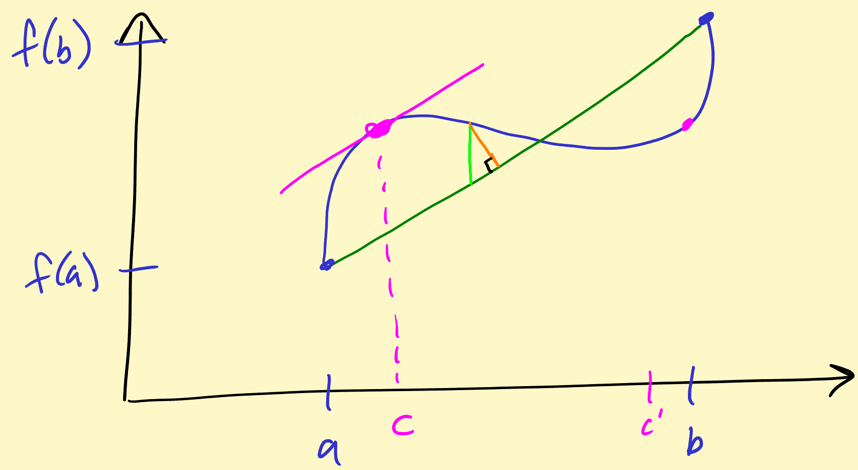
Case $\max = 0$



Mean value theorem: Suppose f continuous on $[a,b]$ and diff'ble on (a,b). There is a c with $a < c < b$ where

$$\underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{average velocity}} = f'(c)$$

instantaneous velocity



| line, | line are proportional,

ratio = $\sin\alpha$

Use Rolle's.

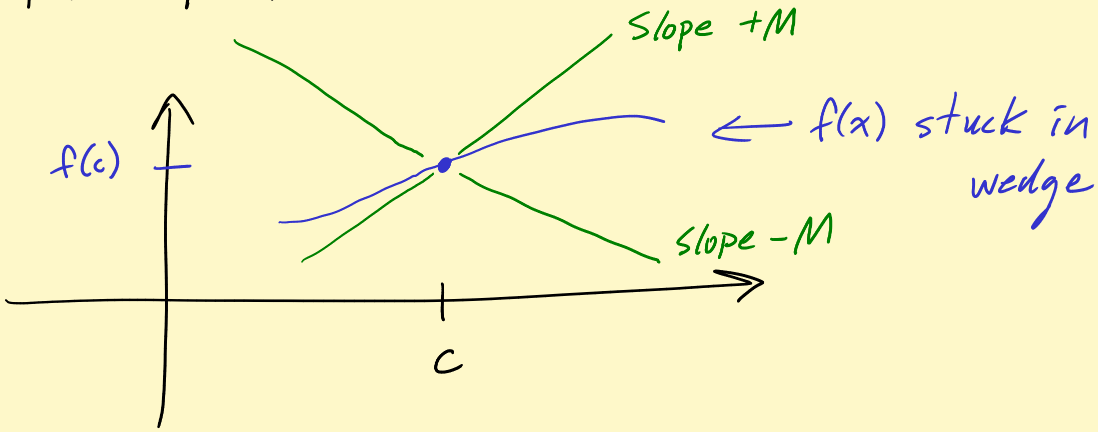
Green line: $f(x) - \left[f(a) + \frac{x-a}{b-a} \cdot \frac{f(b)-f(a)}{b-a} \right]$

MA 341: How do we know what we think we know.

Theorem: If your velocity is zero, you're not moving.

Pf: MVT. $\frac{f(b)-f(a)}{b-a} = \underbrace{f'(c)}_{0}$ $\quad f(b)=f(a)$ ✓

Picture: $|f'(x)| \leq M$



$f(x) \approx \underbrace{f(c) + f'(c)(x-c)}_{\text{best linear approx to } f \text{ near } c}$

Use MVT to estimate error of approx

<u>Darboux's theorem</u> If f is diff'ble on $[a,b]$ and K is between $f'(a)$ and $f'(b)$, then there is a c between a and b with $f'(c)=K$.

<u>Pf</u>: Intermediate value thm? No, because we don't know f' is continuous

Tricky use of MVT.