Lesson 30 L'Hôpital's rule (6.3)

Darboux's theorem f diff'ble on $[a,b]$, K between $f'(a)$ and $f'(b)$. Then there is a c between a and b with $f'(c)=K$.

"Intermediate value theorem for derivatives"

Lemma: $f'>0 \Rightarrow f$ strictly increasing

$f' \geq 0 \Rightarrow f$ increasing

Decreasing cases similar, etc.

Pf: MVT: $x_1 < x_2$: $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$, $x_1 < c < x_2$

Pf of Darboux Case $f'(a) < K < f'(b)$. Let $g(x) = Kx - f(x)$.

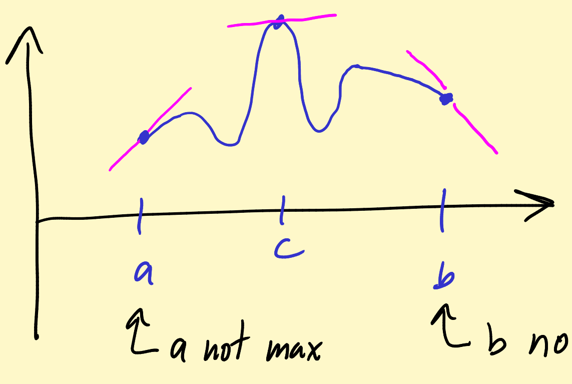
Observe $g'(a) = K - f'(a) > 0 \Leftarrow g$ is strictly increasing at a

$g'(b) = K - f'(b) < 0 \Leftarrow g$ strictly decreasing at b.

Note: g diff'ble $\Rightarrow g$ continuous.

Cont. fcn on closed $[a,b]$ assumes its max value.

Graph of g:



max happens at a c with $a < c < b$

Know, g diff'ble at c and max $\Rightarrow g'(c)=0$.

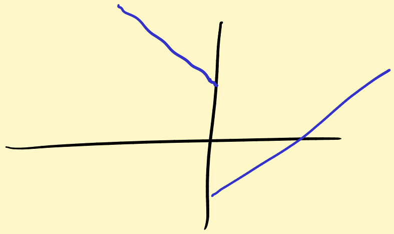
Aha! $g'(c) = K - f'(c) = 0$. So $f'(c) = K$.

Case $f(a) > K > f(b)$ similar

EX: $g(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$

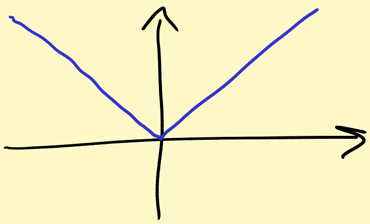
what if $f'(x) = g(x)$? No such thing! Violates Darboux.

$$f(x) = \begin{cases} -x+K & x<0 \\ ? & \\ x+c & x>0 \end{cases}$$



Can make f continuous:

but not diff'ble at $x=0$.



Is there diff'ble f on $[a,b]$ with f' not continuous.

EX: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$ on $[-\pi, \pi]$.

$x \neq 0$: $$f'(x) = \underbrace{2x \sin\frac{1}{x}}_{\to 0 \text{ as } x\to 0} + \underbrace{x^2\left(\frac{-1}{x^2}\right)\cos\frac{1}{x}}_{\text{no limit as } x \to 0}$$

$x=0$: $$DQ = \frac{x^2 \sin\frac{1}{x} - 0}{x} = x\sin\frac{1}{x} \to 0 \text{ as } x \to 0$$

So $f'(0)=0$. f' <u>not</u> continuous at $x=0$.

<u>L'Hôpital's rule</u> L'H: Wrote calc book based on notes for Johann Bernoulli's class!

Baby version $f(x)$, $g(x)$ diff'ble on (a,b), $c \in (a,b)$.

Important: $\begin{cases} f(c)=0 \\ g(c)=0 \end{cases}$

Baby version: $g'(c) \neq 0$.

<u>Fact</u>: $g'(c) \neq 0$ implies there is a $\delta > 0$ such that

$g(x) \neq 0$ if $x \in (c-\delta, c+\delta)$, $x \neq c$. (Easy.)

L'H
$(x \neq c)$
$x \in V_\delta(c)$

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overbrace{f(c)}^{0}}{g(x) - \underbrace{g(c)}_{0}} = \frac{\left[\frac{f(x)-f(c)}{x-c}\right]}{\left[\frac{g(x)-g(c)}{x-c}\right]} \longrightarrow \frac{f'(c)}{g'(c)}$$

Lot's of fancier versions: "Indeterminate forms"

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0

Read 6.3 carefully.

Improvements based on

Extended Mean value theorem (Cauchy's MVT).

Suppose f, g cont on $[a,b]$, diff'ble on (a,b), $g'(x) \neq 0$ for $x \in (a,b)$. Then there is a c between a and b with $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Remark: $g(x) = x$. Get the MVT.

Second baby L'H. If we know f', g' are continuous. $g'(c) \neq 0$.

$$\begin{cases} f(x_0) = 0 \\ g(x_0) = 0 \end{cases} \qquad \frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)}$$

c between x_0 and x. If $x \to x_0$, then $c = c(x) \to x_0$ too. See $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$.

Pf: Pf of Cauchy MVT. (Mysterious!)

$$h(x) := \left(f(x) - f(a)\right) - \frac{f(b)-f(a)}{g(b)-g(a)} \left(g(x) - g(a)\right)$$

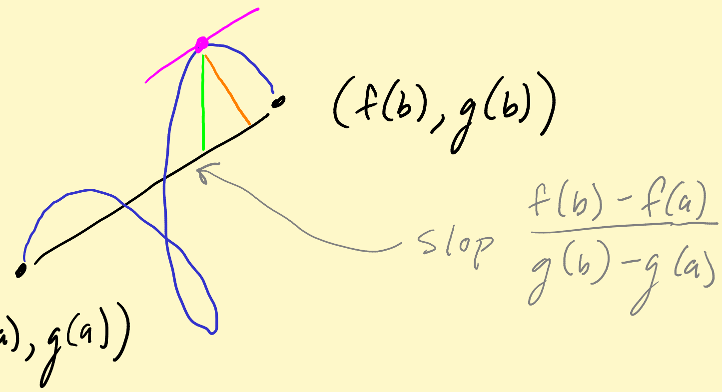
Rolle's: $h(a) = 0$, $h(b) = 0$. Get $a < c < b$

with $h'(c) = 0$

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(c) = 0 \checkmark$$

Where did $h(x)$ come from?! Curve in $\mathbb{R}^2$

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$



$T = f'(c)\hat{\imath} + g'(c)\hat{\jmath}$

Slope : $\frac{f'(c)}{g'(c)}$

$h(x)$ = green line length!

EX: $\frac{1}{x} - \frac{1}{\sin x}$ $\quad x \to 0^+$: $\infty - \infty$

$$\frac{f(x)}{g(x)} = \frac{\sin x - x}{x \sin x} \qquad \frac{0}{0} \qquad x = 0.$$

$$\frac{f'(x)}{g'(x)} = \frac{\cos x - 1}{\sin x + x\cos x} \qquad \frac{0}{0} \qquad x = 0.$$

$$\frac{f''(x)}{g''(x)} = \frac{-\sin x}{2\cos x + x(-\sin x)} \qquad \frac{0}{2} \qquad \text{not indeterminate!}$$

L'H again!

$$0 = \lim_{x\to 0+} \frac{f''(x)}{g''(x)} \overset{L'H}{=} \lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0+} \frac{f(x)}{g(x)}$$