Lesson 31 Taylor's theorem with remainder (6.4)

Classic L'Hôpital's rule examples:

$\ln x$ grows slower than $x^{1/n}$ for any $n \in \mathbb{N}$:

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = 0 \qquad \frac{\infty}{\infty}$$

e^x grows faster than x^n for any $n \in \mathbb{N}$:

$$\lim_{x\to\infty} \frac{e^x}{x^n} = +\infty \quad \text{or} \quad \lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

What theorem do we need?

$$\frac{f(x)}{g(x)} \qquad \frac{f'(x)}{g'(x)}$$

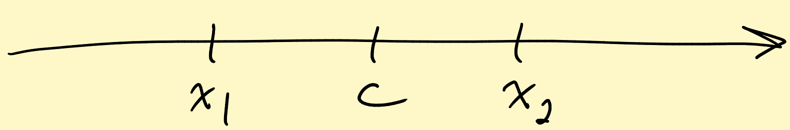
Need $g(x), g'(x)$ nonvanishing for $x > M$.

Need f, g diff'ble

Assume $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$ exists.

Want $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ too.

Tool: $$\frac{f(x_2)-f(x_1)}{g(x_2)-g(x_1)} = \frac{f'(c)}{g'(c)} \qquad \text{(Cauchy MVT)}$$



Big idea: Pick x_1 to make c big enough that

$\dfrac{f'(c)}{g'(c)}$ is as close to L as we want.

x_1 is stuck. Aha! We can let $x_2 \to \infty$.

$$\frac{\frac{1}{g(x_2)}\cdot\left[f(x_2)-f(x_1)\right]}{\frac{1}{g(x_2)}\cdot\left[g(x_2)-g(x_1)\right]} = \frac{f'(c)}{g'(c)} \sim L$$

$$\frac{\dfrac{f(x_2)}{g(x_2)} - \dfrac{f(x_1)}{g(x_2)}}{1 - \dfrac{g(x_1)}{g(x_2)}} \sim L$$

$$\frac{f(x_2)}{g(x_2)} \sim \frac{f(x_1)}{g(x_2)} + L\left(1 - \frac{g(x_1)}{g(x_2)}\right)$$

Aha! Let $x_2 \to \infty$.

Note that $g(x_2) \to \infty$ as $x_2 \to \infty$

$$0 + L\left(1 - 0\right)$$

See $\dfrac{f(x_2)}{g(x_2)} \to L$ as $x_2 \to \infty$.

Start here!

EX: $\displaystyle \lim_{x\to\infty} \frac{x^n}{e^x} \overset{?}{=} \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} \overset{?}{=} \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \overset{?}{=} \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0$

yes! yes yes

EX: $\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\underbrace{\frac{1}{n}x^{\frac{1}{n}-1}}_{\frac{n}{x^{1/n}}}} = 0$ ✓

Taylor's theorem:

MVT shows $f(x) \approx \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{best linear approx to } f(x) \text{ near } x_0}$ for x close to x_0.

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(c)$$

$$f(x) = \underbrace{f(x_0) + f'(c)(x-x_0)}$$ ← c depends on x.

$$\left[f(x) - \left(f(x_0)+f'(x_0)(x-x_0)\right)\right] = \underbrace{\left(f'(c)-f'(x_0)\right)}(x-x_0)$$

Hmmm. If f' is diff'ble, can do MVT again:

$$\frac{f'(c)-f'(x_0)}{x-x_0} = f''(d)$$ d between x_0 and c

$$\left[f(x) - [\text{best linear approx}]\right] = f''(d)(x-x_0)^2$$

$$\left|f(x) - [\text{best linear approx}]\right| \le \left(\max_I |f''|\right)|x-x_0|^2$$

$f(x_0) + f'(x_0)(x - x_0)$ ← has same value as f at x_0 and the same derivative.

$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$ ← agrees with f to order 2 at x_0.

Taylor's theorem

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N$$

$$+ R_N(x)$$

Super duper mean value theorem:

Need $f, f', f'', \ldots, f^{(N)}$ exist and are continuous.

also need $f^{(N+1)}$ to exist.

Then there is a c between x and x_0 such that

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{N+1}$$

Mind blowing fact. Can easily use this to show that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \quad \text{for } \underline{\text{all}}\ x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for } \underline{\text{all}}\ x.$$

Taylor. Get c from a very clever application of Rolle's thm

applied to a very clever choice of fcn! See p. 189.
Short and sweet, but mysterious!

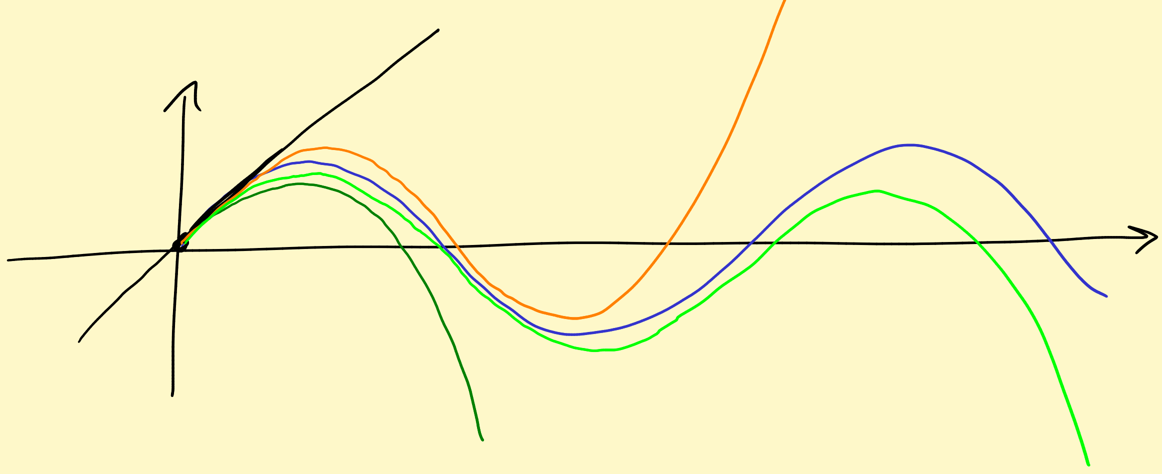
Ex: $f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f^{(4)}(x) = \sin x$ repeat!

$\sin, \cos$ bounded by 1.

$$\left| R_N(x) \right| = \frac{\left| f^{(N+1)}(c) \right|}{(N+1)!} \left| x - x_0 \right|^{N+1}$$

$$\leq \frac{|x-x_0|^{N+1}}{(N+1)!} \longrightarrow 0$$

as $N \to \infty$ for any fixed x, x_0.



Famous fcn. $f(x) = \begin{cases} e^{-1/x^2} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

is C^∞-smooth. f and all its derivatives vanish at 0. $f \neq$ Taylor series