# Lesson 32 The Riemann integral (Read 7.4, 7.1)

Reading: Read Convex fcns, Newton's method in 6.4.

p. 179: 15 Hint: MVT $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$a = x_1$, $b = x_2$

History. Newton, Leibnitz $\leftarrow \int_a^b f\,dx$. Riemann sums.

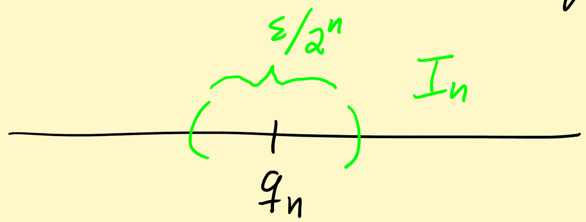
Darboux improved def$^n$s: Darboux integral $\approx$ Riemann integral

Lebesgue: Later. Improved Riemann integral. Invented "measure theory."

EX: $f(x) = \begin{cases} 1 & \text{irrational} \\ 0 & \text{rational} \end{cases}$ $\quad \int_0^1 f(x)\,dx = ?$

$f$ is not Riemann integrable, but it is Lebesgue integrable.

$\int_0^1 f\,dx = 1$ for Lebesgue $\int$.

$\mathbb{Q}$ 


Let $\varepsilon > 0$

$\{q_n\}_{n=1}^{\infty}$ rational #'s in $[0,1]$.

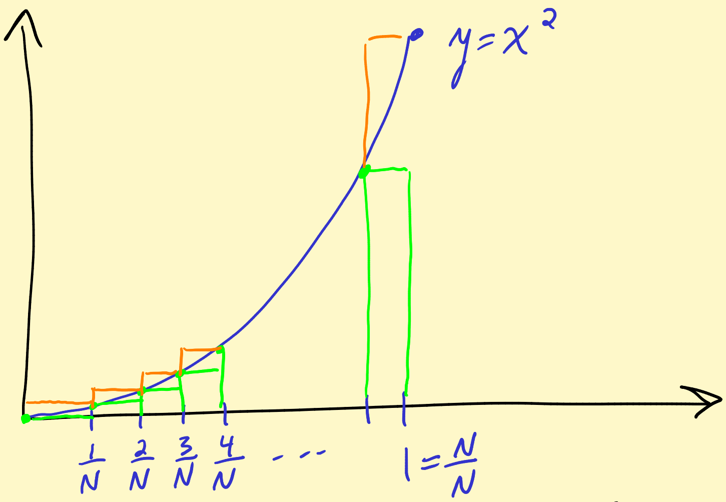
Hmmm. $\mathbb{Q} \cap [0,1] \subset \bigcup_{n=1}^{\infty} I_n \leftarrow$ length $\leq \underbrace{\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}}_{\varepsilon}$

Word: $\mathbb{Q} \cap [0,1]$ is a set of "measure zero."

Complement is a set of "full measure" 1.

Archimedes. Area under a parabola!

EX




$A$ = area under $y = x^2$

$\left( \int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3} \right)$

$$\underbrace{\sum_{n=1}^{N-1} \underbrace{\left(\frac{n}{N}\right)^2}_{f(x_n)} \underbrace{\left[\frac{1}{N}\right]}_{\Delta x}}_{L} < A < \underbrace{\sum_{n=1}^{N} \left(\frac{n}{N}\right)^2 \left[\frac{1}{N}\right]}_{U}$$

$$U - L = \left(\frac{N}{N}\right)^2 \frac{1}{N} = \frac{1}{N}.$$

Expect to "squeeze" $A$.

$$\frac{1}{N^3} \underbrace{\left( \sum_{n=1}^{N-1} n^2 \right)}_{\frac{1}{6}(N-1)N(2(N-1)+1)} < A < \frac{1}{N^3} \underbrace{\left( \sum_{n=1}^{N} n^2 \right)}_{\frac{1}{6}N(N+1)(2N+1)}$$

$$\frac{1}{6}\left(1 - \frac{1}{N}\right)\left(2\left(1-\frac{1}{N}\right) + \frac{1}{N}\right) < A < \frac{1}{6}\left(1 + \frac{1}{N}\right)\left(2 + \frac{1}{N}\right)$$

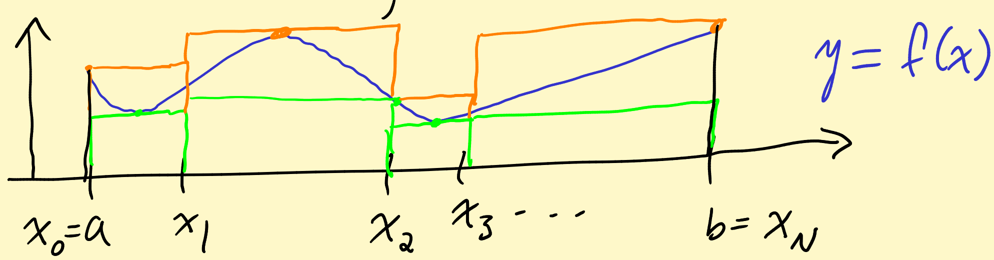
Let $N \to \infty$. Squeeze thm $\frac{1}{3} \le A \le \frac{1}{3}$

Riemann sums. $\int_0^1 f\,dx = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\,\Delta x$ (7.1)

Start with Darboux's version 7.4.

Instead of $\Delta x = \frac{b-a}{N}$, "partition" $[a,b]$ into intervals.

$\mathcal{P}$




Lower sum $= L(\mathcal{P}, f) = \sum_{n=1}^{N} m_n \underbrace{(x_n - x_{n-1})}_{\Delta x}$

$m_n = \inf_{x \in [x_{n-1}, x_n]} f(x)$

Upper sum $= U(\mathcal{P}, f) = \sum_{n=1}^{N} M_n (x_n - x_{n-1})$

$M_n = \sup_{x \in [x_{n-1}, x_n]} f(x)$

Fact: $L(\mathcal{P}_1, f) \le U(\mathcal{P}_2, f)$ for any $\mathcal{P}_1, \mathcal{P}_2$

Def$^n$: 
$$\left.\begin{aligned} L(f) &= \sup_{\mathcal{P}} L(\mathcal{P}, f) \\ U(f) &= \inf_{\mathcal{P}} U(\mathcal{P}, f) \end{aligned}\right\}$$

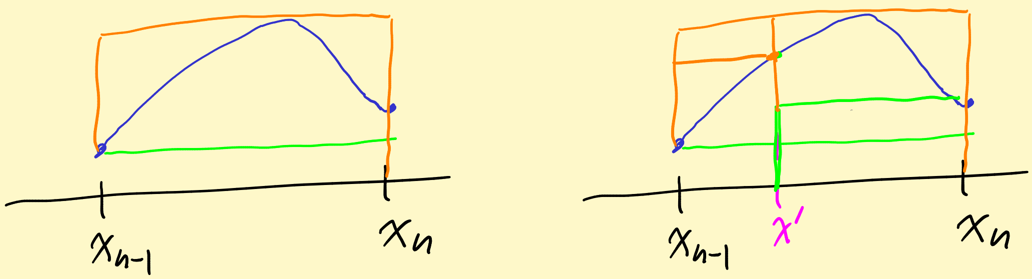
$\boxed{\text{Fact} \quad L(f) \le U(f)}$

Fact: $f$ bounded $\Longrightarrow$ $L(f)$, $U(f)$ exist.

<u>Def$^{n}$</u>: A bounded fcn $f$ is <u>Riemann integrable</u> if $L(f) = U(f)$. Write $\int_a^b f\,dx = L(f) = U(f)$.

<u>Easy facts</u>: Step 1: Have $\mathcal{P}_1$. Get $\mathcal{P}_2$ from $\mathcal{P}_1$ by adding a single pt to $\mathcal{P}_1$.

Claim $$L(\mathcal{P}_1, f) \le L(\mathcal{P}_2, f) \overset{\text{Know}}{\le} U(\mathcal{P}_2, f) \le U(\mathcal{P}_1, f)$$




Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ $\longleftarrow$ tossing in one $x'$ at a time.

$$L(\mathcal{P}_1, f) \le L(\mathcal{P}, f) \le U(\mathcal{P}, f) \le U(\mathcal{P}_3, f)$$

Every lower sum is $\le$ every upper. ✓