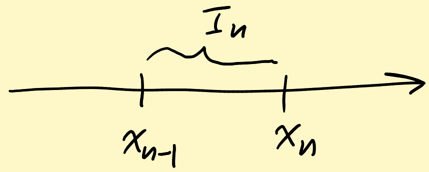
Lesson 33 Basic facts about the Riemann integral

Read 7.4, 7.1, 7.2, 7.3 (in that order)

Last homework, HWK 10 posted now. Due FRIDAY, April 12.

Remark: "The Darboux integral" is the Riemann integral

Partition $\mathcal{P}$:



$$m_n = \inf_{I_n} f \qquad M_n = \sup_{I_n} f$$

$$L(\mathcal{P}, f) = \sum_{n=1}^{N} m_n (x_n - x_{n-1}) \le \sum_{n=1}^{N} M_n (x_n - x_{n-1}) = U(\mathcal{P}, f)$$

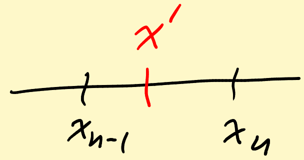
Note: f must be bounded for these to make sense.

Fact: Get $\mathcal{P}'$ from $\mathcal{P}$ by adding a single pt x',

then $$L(\mathcal{P}, f) \le L(\mathcal{P}', f) \le U(\mathcal{P}', f) \le U(\mathcal{P}, f)$$

Repeat, see same true addining m points.

Pf that $U(\mathcal{P}', f) \le U(\mathcal{P}, f)$



$$\ldots + \underbrace{M_1'}_{\le M_n}(x' - x_{n-1}) + \underbrace{M_2'}_{\le M_n}(x_n - x') + \cdots$$

$$\le \ldots + M_n(x' - x_{n-1}) + M_n(x_n - x') + \cdots$$

$$M_n(x_n - x_{n-1}) \qquad\qquad = U(\mathcal{P}, f)$$

U's can only go down. Similarly, the L's can only go up.

Compare $\mathcal{P}_1$ and $\mathcal{P}_2$. Trick: Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$

gotten by tossing in pts in $\mathcal{P}_2$ that are not in $\mathcal{P}_1$

one at a time. Get

$$L(\mathcal{P}_1, f) \le L(\mathcal{P}, f) \le U(\mathcal{P}, f) \le U(\mathcal{P}_2, f)$$

See all L's $\le$ all U's.

So L's bounded above.

and U's bounde below.

$$L(f) = \sup_{\mathcal{P}} L(\mathcal{P}, f) \le \inf_{\mathcal{P}} U(\mathcal{P}, f) = U(f)$$

<u>Def^n</u> A bounded fcn f is Riemann integrable if

$$L(f) = U(f).$$

<u>EX</u>: $h(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$ is <u>not</u> R-int'ble.

$m_n = 0$, $M_n = 1$. $[a, b]$.

$L(\mathcal{P}, h) = 0$. $\quad U(\mathcal{P}, h) = \sum_{n=1}^{N} (x_n - x_{n-1}) = b - a$

EX: Thomé fcn is ! (see book)

$$T(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ (is rational), } m \text{ \& } n \text{ no common factors} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

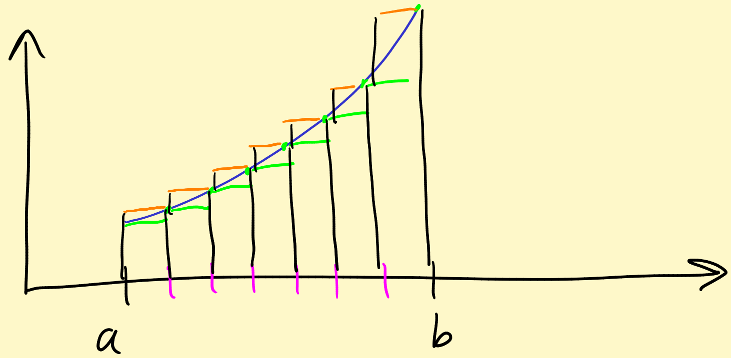
T is continuous at irrational pts. Not at rational.

Hard to believe, but $\int_0^1 T(x)\,dx = 0$.

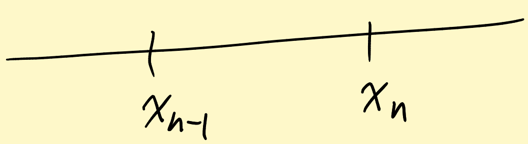
Cauchy criterion If, for every $\varepsilon > 0$, there is a partition $\mathcal{P}$ such $\underbrace{U(\mathcal{P},f) - L(\mathcal{P},f)}_{\geq 0} < \varepsilon$, then $L(f) = U(f)$ and so f is R-int'ble. [f bndd]

Pf: Easy.

Consequence Monotone fcns are R-int'ble.



$$\Delta x = \frac{b-a}{n}$$



$M_n = f(x_n)$

$m_n = f(x_{n-1})$

Let $\varepsilon > 0$.

$$U(\mathcal{P},f) - L(\mathcal{P},f) = \sum_{n=1}^{N} \underbrace{(M_n - m_n)}_{f(x_n)-f(x_{n-1})} \Delta x$$

$$= \Delta x \underbrace{\sum_{n=1}^{N} \left(f(x_n) - f(x_{n-1})\right)}_{f(b)-f(a)}$$

$$= \frac{b-a}{n}\left(f(b)-f(a)\right) < \varepsilon \quad \text{want}$$

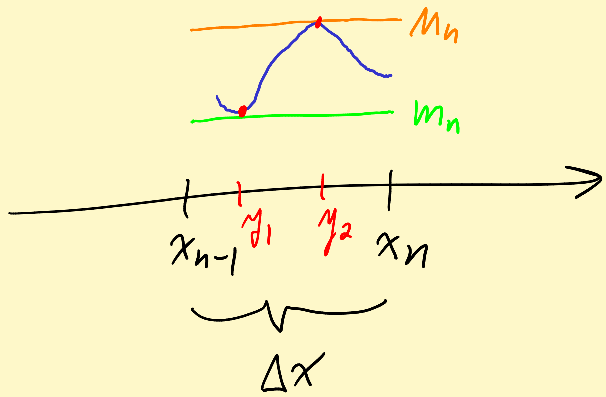
Take $n > \dfrac{(b-a)(f(b)-f(a))}{\varepsilon}$ ✓

<u>Biggy</u> Continuous fcns $\in \mathcal{R}[a,b]$ ← Riemann integrable on $[a,b]$

because f continuous on a closed interval

$\Longrightarrow$ f uniformly continuous.

Do same routine $\Delta x = \dfrac{b-a}{N}$.



Max and mins are assumed!

Let $\varepsilon > 0$.

There is a $\delta > 0$ such that $|y_1 - y_2| < \delta$ implies $|f(y_1) - f(y_2)| < \varepsilon$, $y_1, y_2 \in [a,b]$.

Aha! Want $\dfrac{b-a}{N} < \delta$. Then

$$U(\mathcal{P},f) - L(\mathcal{P},f) = \sum_{n=1}^{N} (M_n - m_n)\Delta x$$

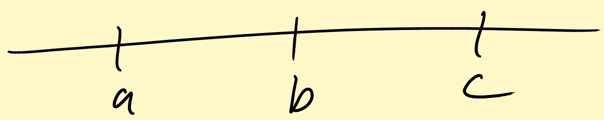
$$M_n - m_n$$

$$= \underbrace{f(y_2) - f(y_1)}_{\geq 0} < \varepsilon$$

$$\leq \underbrace{\sum_{n=1}^{N} \varepsilon \, \Delta x}_{\varepsilon \cdot N \cdot \frac{b-a}{N}} = \varepsilon(b-a)$$

Pretty good. Go back and replace ε by $\varepsilon' = \frac{\varepsilon}{b-a}$ and repeat.

Read book for pfs of easy facts



$$\int_a^c = \int_a^b + \int_b^c$$

$$\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx \qquad k \text{ constant.}$$

$$\int_a^b f + g \, dx = \int_a^b f \, dx + \int_a^b g \, dx$$

$f \leq g$ on $[a,b]$. Then $\int_a^b f\,dx \leq \int_a^b g\,dx$.

$m \leq f \leq M$ on $[a,b]$: $m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$

Famous fact A fcn f on $[a,b]$ is Riemann integrable if and only if

1) f is bounded, and

2) f is continuous "almost everywhere"

[Thomé is.]

"almost everywhere" means on $[a,b]$ minus perhaps a set of "measure zero".

S' "measure zero" means: given $\varepsilon > 0$, there are countably many intervals (a_n, b_n),

$$S' \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$$

$$\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$