Lesson 34 The Fundamental theorem of Calculus (7.3)

If $f, g \in \mathcal{R}[a,b]$ and $f \le g$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Pf

$$L(\mathcal{P}, f) = \sum_{n=1}^{N} m_f^n (x_n - x_{n-1}) \le \sum_{n=1}^{N} m_g^n (x_n - x_{n-1})$$

$$\le \bar{U}(\mathcal{P}, g)$$

Take $\mathcal{P}_1$ making $L(\mathcal{P}_1, f)$ close to $L(f) = \bar{U}(f)$.

Take $\mathcal{P}_2$ making $\bar{U}(\mathcal{P}_2, g)$ close to $L(g) = \bar{U}(g)$.

Squeeze $L(f), L(g)$ using $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

Theorem: If f is continuous on $[a,b]$, and

$$F(x) = \int_a^x f(t)\,dt,$$ then F is diff'ble on

$[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.

Pf: We know cont fcns are in $\mathcal{R}[a,b]$.

We also know, if f is cont on $[a,b]$, then so is $|f|$.

Lemma:

$$\left| \int_a^b f\,dx \right| \le \int_a^b |f|\,dx$$

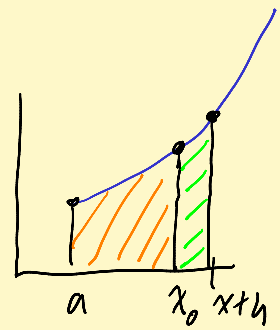
why: $-|f| \le f \le |f|$

So $$\underbrace{-\int_a^b |f|\,dx \le \int_a^b f\,dx \le \int_a^b |f|\,dx}$$

So $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$

Back to FTC pf: $DQ = \dfrac{F(x_0+h)-F(x_0)}{h} =$

$$= \frac{1}{h}\Big[\underbrace{\int_a^{x_0+h} f\,dt - \int_a^{x_0} f\,dt}_{\int_{x_0}^{x_0+h} f\,dt}\Big]$$



Hmmm. $$\frac{1}{h}\int_{x_0}^{x_0+h} \underbrace{f(x_0)}_{\text{constant}}\,dt = \frac{1}{h}\big[f(x_0)\,h\big] = f(x_0)$$

Want $|DQ - f(x_0)| \to 0$ as $h \to 0$.

$$\left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt\right|$$

$$= \left|\frac{1}{h}\int_{x_0}^{x_0+h}\big[f(t)-f(x_0)\big]\,dt\right|$$

$$\leq \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)| \, dt$$

$$\leq \frac{1}{h} \left(\underset{[x_0, x_0+h]}{\mathrm{Max}} |f(t) - f(x_0)| \right) \cdot h$$

Let $\varepsilon > 0$. f cont at x_0. So $\exists\ \delta > 0$ such that $|f(t) - f(x_0)| < \varepsilon$ when $|t - x_0| < \delta$.

So, if $h < \delta$, then $|DQ - f(x_0)| < \varepsilon$. ✓

<u>Or</u> $\quad f(t) = f(x_0) + E(t)$ where $E(t) \to 0$ as $t \to x_0$.

Define $E(x_0) = 0$.

$$f(t) - f(x_0) = E(t)$$

$$|DQ - f(x_0)| \leq \frac{1}{h} \int_{x_0}^{x_0+h} |E(t)| \, dt$$

<u>2^{nd} version of Fund thm</u>: Suppose f is cont on $[a,b]$ and suppose F is an "antiderivative" for f on $[a,b]$, meaning F' exists and $F'(x) = f(x)$ for $x \in [a,b]$.

Then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

Why Suppose $F' = f$.

Version 1: $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$ too.

So $\frac{d}{dx}\left[F(x) - \int_a^x f(t)\,dt\right] \equiv 0$

MVT $\Rightarrow$ (*) $F(x) - \int_a^x f(t)\,dt \equiv C$, a const.

How to figure out what C is: Know

$$\int_a^a f(t)\,dt = 0$$

Plug in $x=a$ in (*): $F(a) - 0 = C$

Now: $F(x) - \int_a^x f(t)\,dt = F(a)$.

Aha! Let $x=b$: $F(b) - \int_a^b f(t)\,dt = F(a)$ ✓

Interesting questions: What if f is <u>not</u> continuous, but $f \in \mathcal{R}[a,b]$. Does Fund thm hold?

If $f \in \mathcal{R}[a,b]$, is $|f| \in \mathcal{R}[a,b]$?

Steps: If $f \in \mathcal{R}[a,b]$, then

$$F(x) = \int_a^x f(t)\,dt \quad \text{is } \underline{\text{continuous}}.$$

Why $\quad \left| F(x) - F(x_0) \right| = \left| \int_{x_0}^x f(t)\,dt \right| \qquad x > x_0$

$$\leq \underbrace{\left(\sup_{[a,b]} |f| \right)}_{= M < \infty} (x - x_0)$$

In fact, F is Lipschitz continuous.

Argument above shows. $F'(x) = f(x)$ at points x where f is continuous.

EX: Thomé fcn $\quad f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is rational} \end{cases}$

no common factors

Cont at irrationals, not at rationals.

Define $F(x) = \int_a^x f(t)\,dt$

Thomé fcn.

Then F is continuous, and F' exists at the irrationals and $F'(x)=0$ if x is irrational.

(In fact, $F \equiv 0$.)

<u>Lebesgue's thm</u> $f \in \mathcal{R}[a,b] \iff$

1) f is <u>bounded</u> on $[a,b]$, and

2) f is <u>continuous</u> <u>almost everywhere</u>.

<u>Cor</u>: If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ too.

<u>why</u>: $|f|$ is cont where f is. f cont a.e. $\Rightarrow$ $|f|$ is too. ✓