Lesson 35 Applications of the integral (7.3, 7.5)

HWK 10 due FRIDAY

Lebesgue's integrability criterion $f \in \mathcal{R}[a,b] \iff$

1) f is bounded

2) f is continuous almost everywhere

$\uparrow$ $[a,b] - S'$, S' is measure zero:

Given $\varepsilon > 0$, $S' \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$ where

$\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$

Fact: $f \in \mathcal{R}[a,b] \Rightarrow |f| \in \mathcal{R}[a,b]$.

Pf: Lebesgue's criterion ✓

Composition theorem: Suppose $f \in \mathcal{R}[a,b]$,

$f([a,b]) \to [c,d]$.

If φ is continuous on $[c,d]$, then $\varphi \circ f \in \mathcal{R}[a,b]$.

Pf: Easy with Lebesgue's C. $\varphi \circ f$ is continuous where f is.

Interesting thing: f, g continuous $\Rightarrow$ fg continuous

$\Rightarrow fg \in \mathcal{R}[a,b]$.

Does $f, g \in \mathcal{R}[a,b] \Rightarrow fg \in \mathcal{R}[a,b]$?

Yes! $x \mapsto x^2$ is continuous on any $[c,d]$.

Composition thm yields that $f, g \in \mathcal{R}[a,b] \Rightarrow$

$$f^2, g^2 \in \mathcal{R}[a,b].$$

Hmmm. $f+g \in \mathcal{R}[a,b]$ too. So $\underbrace{[f+g]^2}_{f^2+2fg+g^2} \in \mathcal{R}[a,b]$.

Aha! $fg \in \mathcal{R}[a,b]$.

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_a^b \underbrace{u(x)}_{G}\,\underbrace{\frac{dv}{dx}(x)}_{f}\,dx = \underbrace{u(x)}_{G}\underbrace{v(x)}_{F}\Big|_a^b - \int_a^b \underbrace{v(x)}_{F}\,\underbrace{\frac{du}{dx}(x)}_{g}\,dx$$

Book $\quad \int_a^b f\,G = FG\Big|_a^b - \int_a^b F\,g \quad$ same thing!

Pf: $(uv)' = uv' + u'v$

$$\underbrace{\int_a^b (uv)'\,dx}_{(uv)\big|_a^b} = \int_a^b u\,\underbrace{v'\,dx}_{dv} + \int_a^b v\,\underbrace{u'\,dx}_{du}$$

<u>Fun fact</u> Fourier coeff $\to 0$ as $n \to \infty$ when f is C^1 smooth (f cont., f' exists and is also cont.)

$$f(x) \underset{=?}{\sim} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

on $[-\pi, \pi]$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Pf that $a_n, b_n \to 0$ as $n \to \infty$ when $f \in C^1[-\pi, \pi]$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{u} \underbrace{\cos nx\, dx}_{dv}$$

$f'(x)dx = du$ $\qquad$ $v = \int \cos nx\, dx = \frac{1}{n} \sin nx$

$$= \underbrace{\left[f(x) \frac{1}{n} \sin nx \right]_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} \frac{1}{n} \sin nx\, f'(x)\, dx$$

$$|a_n| \le \frac{1}{n} \int_{-\pi}^{\pi} \underbrace{|\sin nx|}_{\le 1} |f'(x)|\, dx$$

$$\le \frac{M}{n} \quad \text{where} \quad M = \left(\max_{[-\pi,\pi]} |f'|\right) \underbrace{(2\pi)}_{\text{length } [-\pi,\pi]}$$

$\longrightarrow 0$ as $n \to \infty$. b_n similar.

Famous fact: Riemann-Lebesgue Lemma

$f \in \mathcal{R}[a,b]$ $\quad f \in \mathcal{L}[a,b]$, same.

$\Longrightarrow a_n, b_n \to 0$

Taylor's theorem with remainder in <u>integral form</u>.

Suppose $f', f'', \dots, f^{(n+1)}$ exist and $f^{(n+1)} \in \mathcal{R}[a,b]$.

(feel free to assume $f^{(n+1)}$ is continuous too)

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x) = \dfrac{1}{n!}\displaystyle\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt$.

<u>Pf</u>: Integrate R_n by parts!

$$R_n(x) = \frac{1}{n!}\int_a^x \underbrace{(x-t)^n}_{u} \, \underbrace{f^{(n+1)}(t)\,dt}_{dv}$$

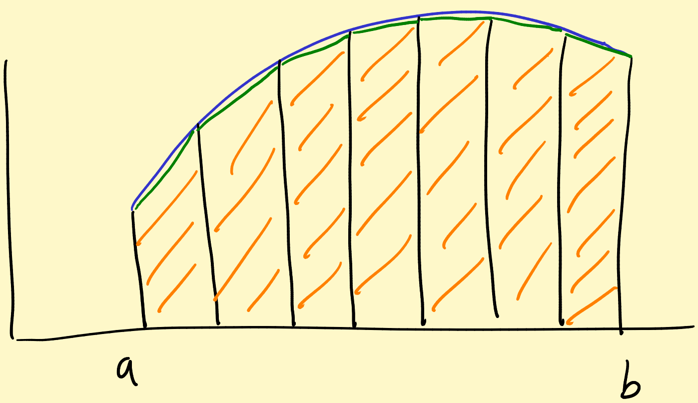
$du = n(x-t)^{n-1}(-1)\,dt \qquad v = f^{(n)}(t)$

$$= -\frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)\,(x-t)^{n-1}\,dt$$

Repeat!

$$= -\frac{f^{(n)}(a)}{n!}(x-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} - \cdots - \underset{\substack{\uparrow \\ \text{stop}}}{f(a)}$$

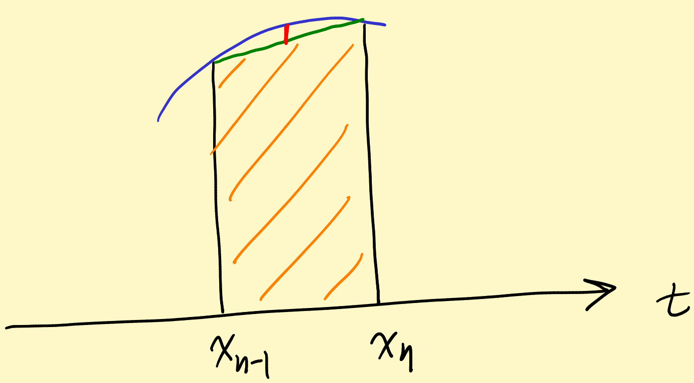
<u>Fun thing</u>



Trapezoid rule (7.5)

How to know when you get $\int_a^b f\,dx$ within some tolerance?

Assume $f \in C^2[a,b]$. (f, f', f'' exist and are cont.)



Error:

$$\left| \int_{x_{n-1}}^{x_n} \Big[\underbrace{f(t) - (f(x_{n-1})}_{f'(c_n)(t-x_{n-1})} + \frac{f'(x_{n-1})+f'(x_n)}{2}\underbrace{(t-x_{n-1})}_{<\Delta x}) \Big] dt \right.$$

$$\frac{2f'(c_n) - f'(x_{n-1}) - f'(x_n)}{2} \underbrace{(t - x_{n-1})}_{< \Delta x}$$

$$\frac{[f'(c_n) - f'(x_{n-1})] + [f'(c_n) - f'(x_n)]}{2}$$

Use MVT again!

$$|\quad| \le \frac{|f''(c_n')(c_n - x_{n-1})| + |f''(c_n'')(x_n - c_n)|}{2}$$

Let $M = \underset{[a,b]}{\text{Max}} |f''|$. $\quad \le \dfrac{M\Delta x + M\Delta x}{2} \qquad \Delta x = \dfrac{b-a}{N}$

$$\text{Error: } \le \int_{x_{n-1}}^{x_n} M \frac{(b-a)}{N} \frac{(b-a)}{N}\, dt$$

$$\le M\left[\frac{(b-a)}{N}\right]^2 \cdot \frac{(b-a)}{N}$$

Have N of these! $\qquad$ Total Error $< \dfrac{M(b-a)^3}{N^2}$