Lesson 36 Riemann's Riemann integral (7.1)

Change of variable formula:

$$\int \sin(x^2)\, x\,dx = \frac{1}{2}\int \sin(\underset{\uparrow\, u}{x^2})\,\underbrace{(2x)\,dx}_{\underbrace{\frac{du}{dx}\,dx}_{du}}$$

$$= \frac{1}{2}\int \sin u\,du = \frac{1}{2}(-\cos u) + C$$

$$= -\frac{1}{2}\cos x^2 + C$$

Thm: $$\int_a^b f(u(x))\,\frac{du}{dx}\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Pf: Fund thm calc plus chain rule.

Let $F(x)$ be such that $F'(x) = f(x)$.

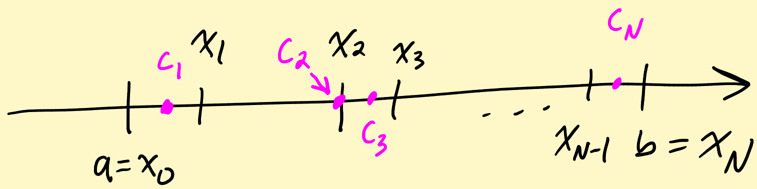
Then $$\int_a^b f(u(x))\,\frac{du}{dx}\,dx = \int_a^b \underbrace{F'(u(x))\,\frac{du}{dx}}_{\frac{d}{dx}F(u(x))}\,dx$$

$$= F(u(x))\Big|_a^b$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} f(u)\,du \quad \checkmark$$

Riemann's $\int$



$\wp$

$c_n \in [x_{n-1}, x_n]$ $\qquad$ $\mathcal{P}$ partition

$\dot{\mathcal{P}}$ "tagged" partition

$$\text{"Mesh"} = \|\dot{\mathcal{P}}\| = \underset{n=1,\dots,N}{\operatorname{Max}} (x_n - x_{n-1}) \leftarrow \text{biggest } \Delta x$$

Riemann sum: $\quad S(\dot{\mathcal{P}}, f) = \sum_{n=1}^{N} f(c_n)(x_n - x_{n-1})$

Def^n: If f is bounded on $[a,b]$, we say f is Riemann integrable and write $f \in \mathcal{R}[a,b]$ and say $\int_a^b f = \int_a^b f(x)\,dx = L$ if, for any $\varepsilon > 0$, no matter how small, there is a $\delta > 0$, which might depend on ε, such that

$$|S(\dot{\mathcal{P}}, f) - L| < \varepsilon$$

whenever $\dot{\mathcal{P}}$ is a tagged partition of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \delta$.

Fact: Equivalent to Darboux's def^n in 7.4.

Pf: p. 232.

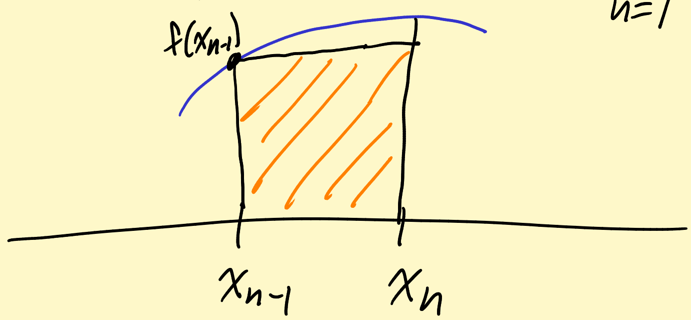
Last time: $\left| \int_a^b f(x)\,dx - (\text{Trapezoid rule sum}) \right|$

$$\leq \frac{M(b-a)^3}{N^2}$$

where $M = \underset{[a,b]}{\mathrm{Max}} |f''|$. $\Delta x = \frac{b-a}{N}$.

Ques How good is the rectangle method?

EX: Left endpoints. $I \approx \sum_{n=1}^{N} f(x_{n-1})\,\Delta x$, $\Delta x = \frac{b-a}{N}$



Error: $\displaystyle \int_{x_{n-1}}^{x_n} f(x)\,dx - \underbrace{f(x_{n-1})\,\Delta x}_{\int_{x_{n-1}}^{x_n} f(x_{n-1})\,dx}$

$$= \int_{x_{n-1}}^{x_n} \underbrace{\left[f(x) - f(x_{n-1}) \right]} \, dx$$

hmmm. $\frac{f(x) - f(x_{n-1})}{x - x_{n-1}} = f'(c)$ MVT

$$f(x) - f(x_{n-1}) = f'(c)(x - x_{n-1})$$

Aha! So $|f(x)-f(x_{n-1})| = \underbrace{|f'(c)|}_{\le M} \underbrace{(x-x_{n-1})}_{\le \Delta x}$

where $M = \underset{[a,b]}{\mathrm{Max}}|f'|$

So $\left| \int_{x_{n-1}}^{x_n} f\,dx - f(x_{n-1})\Delta x \right|$

$$\le \int_{x_{n-1}}^{x_n} \underbrace{|f(x)-f(x_{n-1})|}_{\le M\Delta x}\,dx \le (M\Delta x)\underbrace{(x_n - x_{n-1})}_{\Delta x}$$

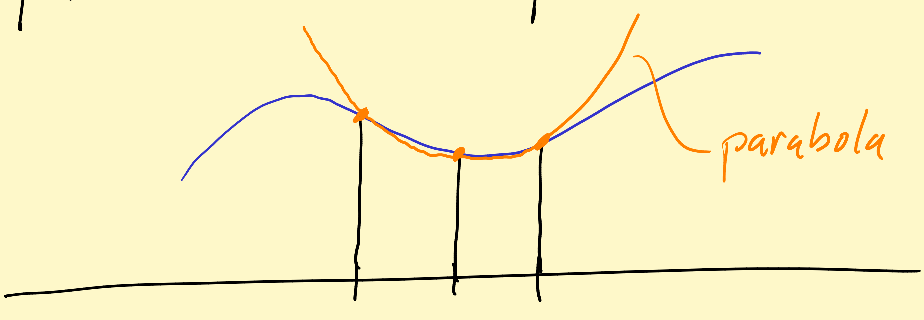
$$\left| \int_a^b f(x)\,dx - (\text{Rec method}) \right| = \sum^{N} \left(\text{Local errors}\right)$$

$M\left(\frac{b-a}{N}\right)^2$

$$\le \frac{M(b-a)^2}{N}$$

much worse than T. rule!

Super duper method: Simpson's rule.



even partition

Simpson's rule error: $\frac{(b-a)}{180}\left(\frac{b-a}{N}\right)^4 \underbrace{\underset{[a,b]}{\mathrm{Max}}\left|f^{(4)}\right|}$

(7.5) Fancy MVT $\longrightarrow$ $= f^{(4)}(c)$, $c \in (a,b)$

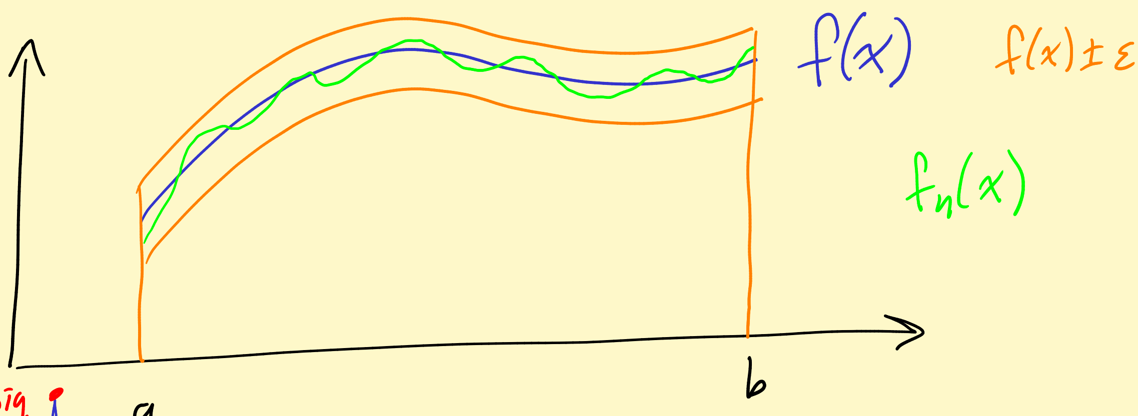
Important subjects in Chap 8

Uniform convergence of functions on $[a,b]$:

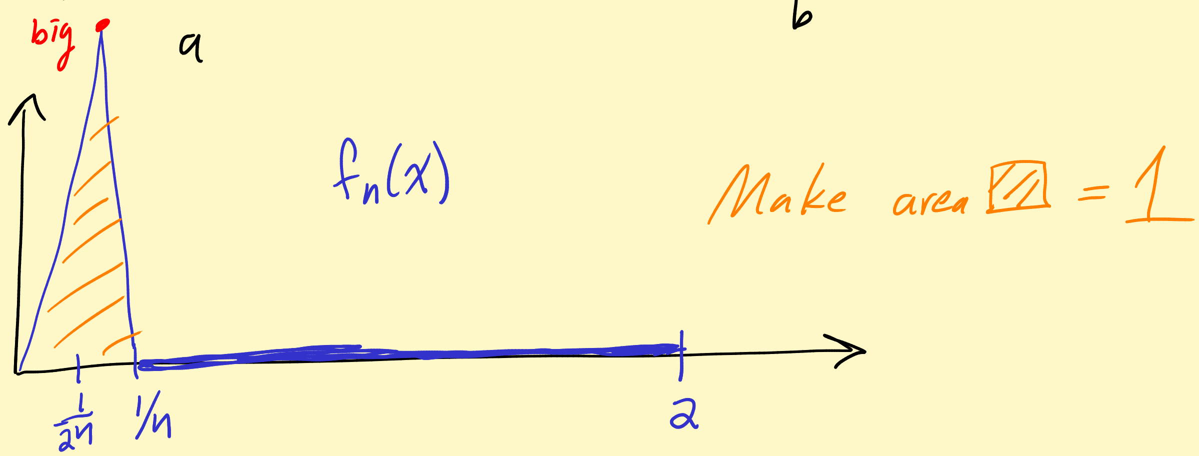
$f_n(x)$ converges to $f(x)$ uniformly on $[a,b]$ means, given $\varepsilon > 0$, there is a pos integer N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ when $n > N$.

Pointwise convergence means $f_n(x) \to f(x)$ for each $x \in [a,b]$.

Important fact Uniform limits of continuous functions are continuous.



EX:



$f_n(x) \longrightarrow$ pointwise to the zero fcn.

but, f_n are not even bounded.

And $\int_0^2 f_n \, dx$ do not tend to zero.

Weierstraß M-test. Suppose f_n are continuous on $[a,b]$ and $|f_n(x)| \le M_n$. If $\sum_{n=1}^{N} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly to a on $[a,b]$ to a continuous fcn on $[a,b]$.