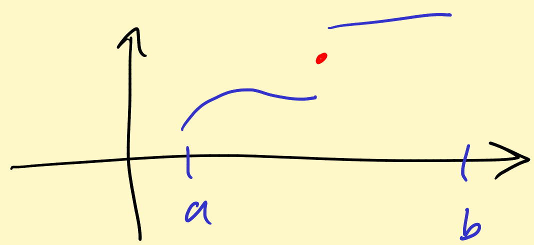
Lesson 37 Sequences of functions (8.1, 8.2)

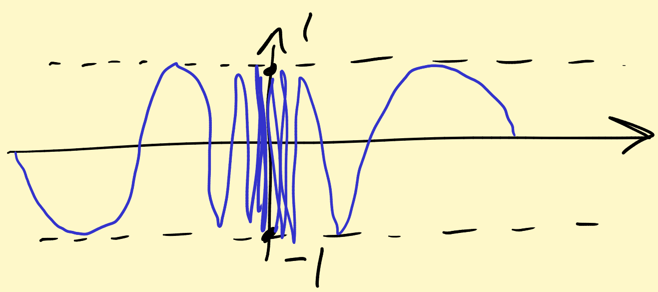
Fancier versions of the fundamental theorem of calculus:

Fact: Suppose f is bounded on $[a,b]$ and f is continuous on $[a,b]$ except at perhaps finitely many pts $E \subset [a,b]$. Then $f \in \mathcal{R}[a,b]$.

Ex:



Ex:

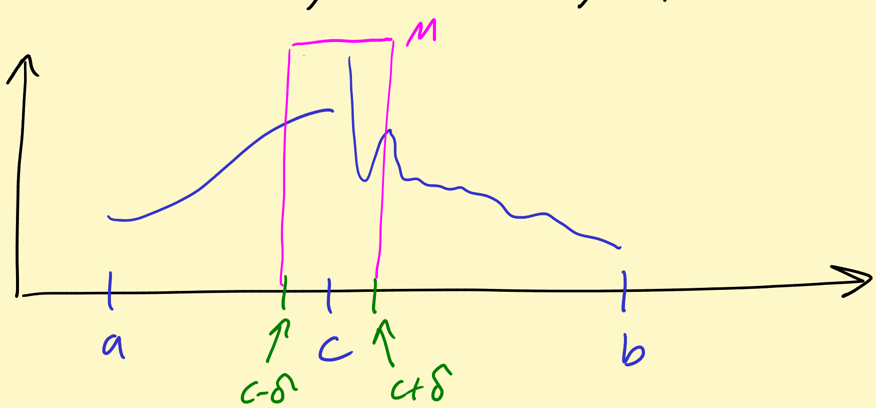


$\sin \frac{1}{x}$ cont on $[-\pi, \pi] - \{0\}$. Bndd by 1.

Pf of fact Enough to add a single pt of discont.

Suppose $|f(x)| \le M$, f discontinuous at c.

Let $\varepsilon > 0$. Use Cauchy criterion: Find partition $\mathcal{P}$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$.



Plan: Get upper sums and lower sums.

$$L_a^{c-\delta} \le \int_a^{c-\delta} f\,dx \le U_a^{c-\delta}$$

differ by $< \varepsilon/3$

same for $\int_{c+\delta}^{b}$

$$-M2\delta < \Big(\inf_{[c-\delta,c+\delta]} f\Big)2\delta \le \Big(\sup_{[c-\delta,c+\delta]} f\Big)2\delta < M2\delta$$

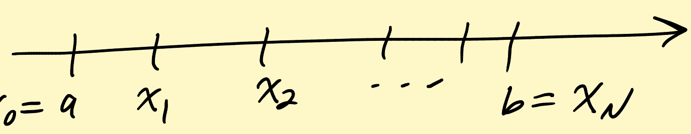
Want $\underbrace{(M2\delta)-(-M2\delta)}_{4\delta M} < \frac{\varepsilon}{3}$

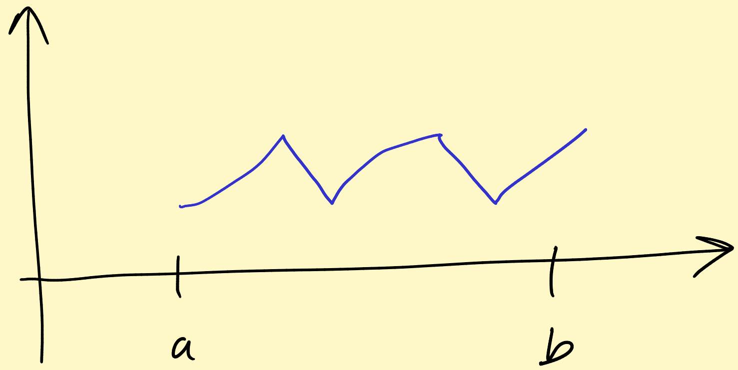
Aha! Take $\delta = \dfrac{\varepsilon/3}{4M}$.

Fancier version of the FTC: If f is "piecewise $\mathcal{C}^1$-smooth" on $[a,b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Defn: Piecewise smooth allows "kinks, but no jumps".





$f(x)$ continuous on $[x_{n-1}, x_n]$ and f' exists on $[x_{n-1}, x_n]$.

Values of f hook up: $\lim\limits_{x \to x_n^-} f(x) = \lim\limits_{x \to x_n^+} f(x)$,

but values of f' don't have to hook up.

Pf: $$\int_a^b f' \, dx = \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} f' \, dx$$

$$= \sum_{n=1}^{N} \left[f(x_n) - f(x_{n-1}) \right]$$

$$= \left(\cancel{f(x_1)} - f(x_0) \right) + \left(\cancel{f(x_2)} - \cancel{f(x_1)} \right) + \left(f(x_3) - \cancel{f(x_2)} \right)$$

$$+ \cdots + \left(\cancel{f(x_{N-1})} - f(x_{N-2}) \right)$$

$$+ f(x_N) - \cancel{f(x_{N-1})}$$

$$= f(x_N) - f(x_0) \checkmark$$

Chap 8: 8.1–8.3 Sequences of functions.

Exciting moment: Can think about functions as "points" in a space!

$C[a,b] = \{\text{continuous fcns on } [a,b]\}$.

$$\text{dist}(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$$

Uniform convergence: $f_n \twoheadrightarrow f$ on $[a,b]$ means, given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in [a,b]$$

when $n \geq N$.

Note: Same as $\text{dist}(f_n, f) < \varepsilon$.

"$f_n \to f$ in $C[a,b]$ as a sequence of pts using the metric $\text{dist}(f_n, f)$."

Def^n: (f_n), $f_n \in C[a,b]$ is called "uniformly Cauchy" if, given $\varepsilon > 0$, there is an N such that

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } x \in [a,b]$$

when $n, m > N$.

Big fact #1 Uniformly Cauchy sequences of fcns in $C[a,b]$ converge uniformly.

Why: Suppose f_n is uniformly Cauchy. Then $(f_n(x))$ is a Cauchy seq for each x.

$\mathbb{R}$ is complete. So $f_n(x)$ converges to something, call it $f(x)$.

Claim: $f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$. Pick $x \in [a,b]$.

Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that

$$|f_n(y) - f_m(y)| < \frac{\varepsilon}{2} \quad \text{when} \quad n,m \geq N,\ y \in [a,b].$$

$$|f(x) - f_n(x)| = |f(x) - f_m(x) + f_m(x) - f_n(x)|$$

$$\leq \underbrace{|f(x) - f_m(x)|}_{\substack{\text{Take gigantic } m \\ \text{so that } |f(x)-f_m(x)| < \frac{\varepsilon}{2} \\ \text{too.}}} + \underbrace{|f_m(x) - f_n(x)|}_{\frac{\varepsilon}{2}}$$

$< \varepsilon$ ✓

Get $|f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ when $n \geq N$.

x was arbitrary! Get $f_n \twoheadrightarrow f$ on $[a,b]$.

<u>Big fact #2</u> Limit fcn f must be cont. too.

Pick $c \in [a,b]$.

$$|f(x) - f(c)| =$$

$$|f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)|$$

$$\leq \underbrace{|f(x) - f_n(x)|}_{\substack{< \frac{\varepsilon}{3} \\ \text{by unif conv.} \\ \text{if } n \geq N}} + |f_n(x) - f_n(c)| + \underbrace{|f_n(c) - f(c)|}_{\substack{\text{unif conv, } n \geq N \\ < \frac{\varepsilon}{3}}}$$

Aha! Let $n = N$.

Know f_N is <u>continuous</u>. Get middle $|\ \ | < \frac{\varepsilon}{3}$ too

when $|x - c| < \delta$. ✓