Lesson 38 Uniform limits of continuous functions

Exam 2 due FRIDAY, 11:00 pm in Gradescope

$C[a,b]$ = continuous functions on $[a,b]$

$$\text{dist}(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| = \max_{x\in[a,b]} |f(x)-g(x)|$$

Book: $= \|f-g\|_{[a,b]}$

(f_n) a seq in $C[a,b]$. $f_n \twoheadrightarrow f$ means $f_n \to f$ uniformly on $[a,b]$.

Big facts: 1) $f_n \twoheadrightarrow f$, then $f \in C[a,b]$.

2) (f_n) uniformly Cauchy $\iff$ (f_n) converges uniformly (to an $f \in C[a,b]$).

3) $f_n \twoheadrightarrow f$, then $\int_a^b f_n\,dx \to \int_a^b f\,dx$ as $n \to \infty$.

4) Weierstraß M-test. $f_n \in C[a,b]$, $|f_n(x)| \le M_n$ on $[a,b]$.

If $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{N} f_n(x)$ converges uniformly (to $S(x) \in C[a,b]$).

Remarks: ($\Rightarrow$) for (2) last time. ✓

($\Leftarrow$) easy: $f_n \twoheadrightarrow f$ then

$$|f_n(x)-f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$$

$$\le |f_n(x)-f(x)| + |f_m(x)-f(x)|$$

Pf of #3: $\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$

$$= \left| \int_a^b \big(f_n(x) - f(x)\big)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx$$

$$\le \left(\max_{x \in [a,b]} |f_n(x) - f(x)| \right)(b-a)$$

$$\longrightarrow 0 \quad \text{as } n \to \infty \text{ if } f_n \rightrightarrows f \text{ on } [a,b].$$

Pf of Weier M-test $\quad S_N(x) = \sum_{n=1}^{N} f_n(x) \qquad N > M$

Then $\left| S_N(x) - S_M(x) \right| = \left| \sum_{n=N+1}^{M} f_n(x) \right|$

$$\le \sum_{n=N+1}^{M} \underbrace{|f_n(x)|}_{\le M_n} \le \sum_{N+1}^{M} M_n$$

Fact: Tail ends of convergent series $\to 0$.

$$\sigma_N = \sum_{n=1}^{N} M_n \longrightarrow L = \sum_{n=1}^{\infty} M_n$$

$$\underbrace{L - \sigma_N}_{\to 0,\ N \to \infty} = \sum_{N+1}^{\infty} M_n \longleftarrow \text{So this} \to 0 \text{ too.}$$

So $0 \leq \underbrace{\sum_{n=N+1}^{M} M_n}_{\substack{\to 0 \\ \text{as } N,M \to \infty \\ \text{(squeeze)}}} \leq \sum_{N+1}^{\infty} M_n$

$\sum_{N+1}^{\infty} M_n \to 0$ as $N \to \infty$

Application of M-test: Power series have a "radius of convergence". Get <u>uniform</u> convergence on

$$[a,b] \subset (x_0 - R, x_0 + R), \quad R > 0$$

R. of C.

8.4 Exponential function e^x.

e^x is unique diff'ble fcn $E(x)$ such that

1) $E'(x) = E(x)$

2) $E(0) = 1$.

All basic properties follow from 1, 2.

Define $E_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$

<u>Claim</u>: $(E_n(x))$ is uniformly Cauchy on $[-R, R]$

<u>Why</u> $|E_N(x) - E_M(x)| = \left| \sum_{n=N+1}^{M} \frac{x^n}{n!} \right|$

$$\leq \sum_{n=N+1}^{M} \frac{|x|^n}{n!}$$

$$\leq \sum_{n=N+1}^{M} \frac{R^n}{n!}$$

$$\left(\frac{R^{N+1}}{(N+1)!} + \cdots + \frac{R^M}{M!} \right)$$

$$\frac{R^{N+1}}{(N+1)!} \underbrace{\left[1 + \underbrace{\frac{R}{N+2}}_{< \frac{R}{N}} + \underbrace{\frac{R^2}{(N+2)(N+3)}}_{< \left(\frac{R}{N}\right)^2} + \cdots + \underbrace{\frac{R^{M-N-1}}{(N+2)\cdots M}}_{< \left(\frac{R}{N}\right)^{M-N-1}} \right]}$$

Want $N > R$:

Really want $N > 2R$.

Geom series $< \dfrac{1}{1-\left(\frac{R}{N}\right)} < 2$

Get $\left| E_N(x) - E_M(x) \right| < \dfrac{R^{N+1}}{(N+1)!} \cdot 2$ when $x \in [-R, R]$, $N > 2R$.

$\longrightarrow 0$ as $N \to \infty$.

$E_N(x)$ unif Cauchy $\Rightarrow$ E_N converges uniformly on $[-R, R]$ to cont fcn $E(x)$.

Notice that $\displaystyle\int_0^x E_n(t)\,dt = \sum_{k=0}^{n} \int_0^x \frac{t^k}{k!}\,dt$

$$= \sum_{k=0}^{n} \frac{x^{k+1}}{(k+1)!}$$

$$= x + \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)!}$$

Aha! Add 1 to both sides. Get

$$\boxed{1 + \int_0^x E_n(t)\,dt = E_{n+1}(x)}$$

Let $n \to \infty$: (Use #3 on left)

$$1 + \int_0^x E(t)\,dt = E(x)$$

Fund thm calc: $\frac{d}{dx}$(left side) exists and is equal to $0 + E(x)$.

Get: 1) $E(x)$ is diff'ble and $E'(x) = E(x)$.

2) $E(0) = 1 + 0 + \frac{0^2}{2!} + \cdots = 1$ ✓

<u>Lemma</u> If $f(x)$ is diff'ble and $f'(x) = k f(x)$, then $f(x) = C\,E(kx)$ where $C = f(0)$.

Pf: Next time.

Consequence. Let $f(x) = E(x+y)$. [$y = \text{const}$]

$$f'(x) = \underbrace{E'(x+y)}_{E(x+y)} \underbrace{\frac{d}{dx}(x+y)}_{(1+0)}$$

$$= 1 \cdot f(x).$$

Lemma says $\underbrace{f(x)}_{E(x+y)} = \underset{E(y)}{C}\, \underbrace{E(1\cdot x)}_{E(x)}$ where $C = f(0)$

$f(0) = E(0+y)$
$= E(y)$

Aha! $E(x+y) = E(x)\,E(y)$ ✓