Lesson 39 The exponential function (8.3)

Exam 2 due in GS by 11:00pm tonight.

Last time: $E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

Showed $E_n(x)$ are uniformly Cauchy on $[-R,R]$.

So $E_n(x)$ converge uniformly on $[-R,R]$ to a <u>continuous</u> fcn $E(x)$.

$$E_{n+1}(x) = 1 + \int_0^x E_n(t)\,dt$$

Unif conv $\Rightarrow$ $$E(x) = 1 + \int_0^x E(t)\,dt$$ RHS is diff'ble

Conclude that $E(x)$ is diff'ble <u>and</u> FTC shows

that $$E'(x) = 0 + E(x).$$

Basic facts $$E_n(x) = 1 + x + \underbrace{\frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}_{>0 \text{ if } x>0.}$$

So $$E_n(x) > 1 + x \quad \text{if } x > 0.$$

$$\Rightarrow \quad E(x) \geq 1 + x \quad \text{if } x > 0.$$

<u>Basic facts</u> 1) $E'(x) = E(x)$

2) $E(0) = \underline{1}$ $\leftarrow$ $E_n(0) = 1$ for all n.

3) $E(x) \geq 1 + x$ if $x > 0$, so

$$\lim_{x\to\infty} E(x) = \infty.$$

4) $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

<u>Lemma</u> If f is diff'ble on $\mathbb{R}$ and

$$f'(x) = k\, f(x) \text{ for all } x,$$

then $f(x) = C\,E(kx)$ where $C = f(0)$.

<u>Pf</u>: (MA 262, 266, or 366 : Linear 1st order ODE)

$f' - kf = 0$ ⟵ want integrating factor u to make LHS $= (\ )'$

$u[f' - kf] = 0$

$$\left.\begin{aligned} uf' - kuf &= 0 \\ [uf]' = uf' + u'f &= 0 \end{aligned}\right\}$$

Need $u' = -ku$.

Have such a thing!

$\boxed{u = E(-kx)}$

$u' = E'(-kx)\frac{d}{dx}[-kx]$

$= E(-kx)\cdot(-k)$ ✓

Do it: $f' - kf = 0$ ⟵ mult by $E(-kx)$

$$E(-kx)f'(x) - kE(-kx)f(x) = 0$$

$$\left[E(-kx)f(x)\right]' \equiv 0$$

Conclude that $\boxed{E(-kx)f(x) \equiv c,}$ a const. *

Plug in $x=0$ to see $\underbrace{E(0)}_{1}f(0)=c$

So $c=f(0)$ ✓

Stop the proof! Start over using $k=1$, $f(x)=E(x)$.

$f'(x)=E'(x)=E(x)=f(x)$ ✓

$f(0)=E(0)=1$. So $c=1$.

Get a new (*): $E(-1\cdot x)\underbrace{f(x)}_{E(x)}=c=1$

Aha! $\boxed{E(-x)E(x) \equiv 1}$

Basic facts: 5) $E(x)$ is never zero, and

6) $E(-x) = 1/E(x)$.

Go back and continue the pf:

(*) $\underbrace{E(-kx)}_{=1/E(kx)}f(x) = f(0)$

Done! $f(x) = f(0)\,E(kx)$ ✓

Basic fact 7) $E(x+y) = E(x)E(y)$

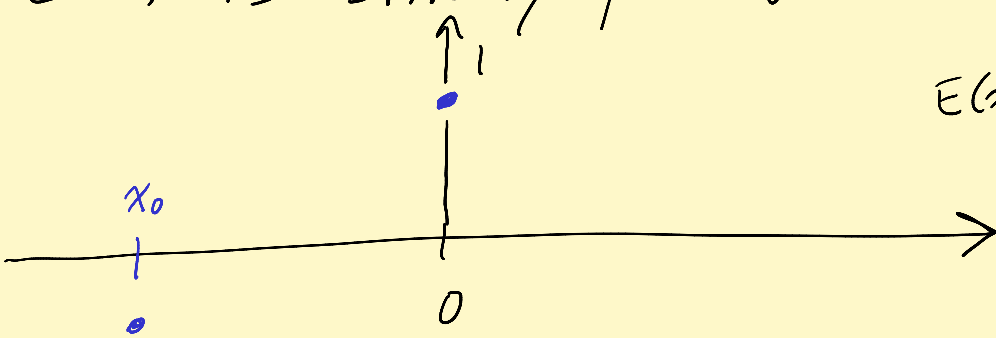
Pf: Lemma: $f(x) = E(x+y)$ $\leftarrow$ $f(0)=E(y)$, $y=$ const.

$$f'(x) = E'(x+y)\frac{d}{dx}(x+y) = E(x+y)\cdot 1$$
$$= 1\cdot f(x) \checkmark$$

Get $\underbrace{f(x)}_{E(x+y)} = \underbrace{f(0)}_{E(y)} E(1\cdot x)$

$$E(x+y) = E(x)E(y) \checkmark$$

9) $E(x)$ is strictly positive.



$E(x)$ is never zero

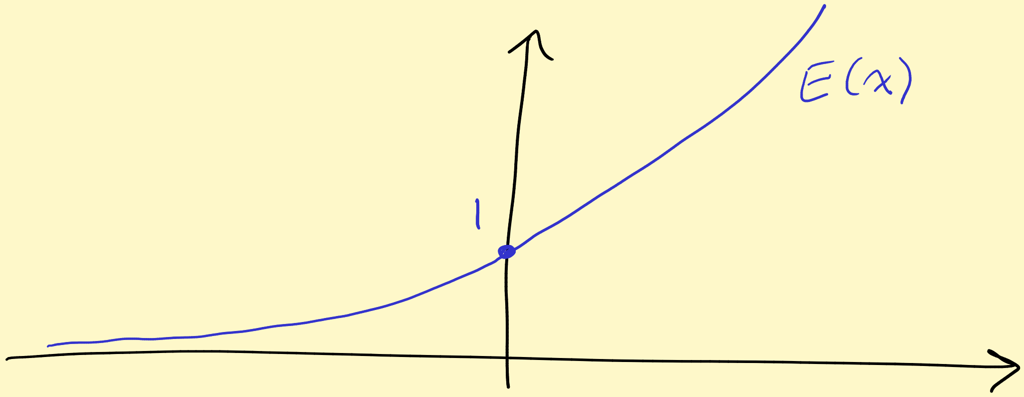
negative here. IMT $\Rightarrow$ $E(x)$ has a zero between x_0 and 0. ↯

10) $E(-x) = \dfrac{1}{E(x)} \longrightarrow 0$ as $x \to \infty$.

$$\lim_{x\to-\infty} E(x) = 0.$$

11) $E(x)$ is strictly increasing

Pf $E'(x) = E(x) > 0$. MVT $\checkmark$



$$E : (-\infty, \infty) \xrightarrow[\text{onto}]{1-1} (0, \infty)$$

12) E is C^∞-smooth : $\left(E' = E,\ E'' = E',\ E''' = E'', \ldots \right)$

Inverse fcn thm yields $L = E^{-1} : (0,\infty) \xrightarrow[\text{onto}]{1-1} (-\infty,\infty)$

L diff'ble and Chain rule yields

$$E(L(x)) = x$$

$$\underbrace{E'(L(x))}_{\underbrace{E(L(x))}_{x}} L'(x) = 1 \qquad \boxed{L'(x) = \frac{1}{x}}$$

$$\begin{cases} L'(x) = \frac{1}{x} \\ L(1) = 0 \end{cases} \qquad L(x) = \int_1^x \frac{1}{t}\, dt$$

$$L(y_1 y_2) = L(y_1) + L(y_2) \qquad y_1, y_2 > 0$$

$y_1 = e^{x_1}$

$y_2 = e^{x_2}$

$$\underbrace{y_1 y_2 = e^{x_1} e^{x_2} = e^{x_1 + x_2}}$$

So $L(y_1 y_2) = \underset{L(y_1)}{x_1} + \underset{L(y_2)}{x_2}$ ✓

Follows from $\begin{cases} e^x \to \infty \text{ as } x \to \infty, \text{ and} \\ e^x \to 0 \text{ as } x \to -\infty \end{cases}$

that $\begin{cases} L(x) \to \infty \text{ as } x \to \infty \\ L(x) \to -\infty \text{ as } x \to 0+ \end{cases}$

For rest of life, write $E(x) = e^x$, $L(x) = \ln x$.

$E(x+y) = E(x)\,E(y)$ ← E is a "group homomorphism" of the additive group of $\mathbb{R}$ to the multiplicative group of $\mathbb{R} - \{0\}$. $L = E^{-1}$

Question: What is $\pi^{\sqrt{2}}$?

$$E(\underbrace{x + x + \cdots + x}_{p\text{-times}}) = E(x) \cdot \cdots \cdot E(x) = E(x)^p$$

So $E(px) = E(x)^p$

$E(px)^{1/p} = E(x)$

$$L(x^r) = r \ln x \qquad (x > 0). \quad r \text{ rational.}$$

$$x = E(L(x))$$

$$x^{p/q} = E(L(x^{p/q})) = E\left(\frac{p}{q} L(x)\right)$$

$$x^{p/q} = e^{\frac{p}{q} \ln x}, \quad \frac{p}{q} \in \mathbb{Q}$$

Forced, by continuity, to define

$$x^r = e^{r \ln x} \quad \text{when } r \text{ is } \underline{\text{irrational}}.$$

$$\pi^{\sqrt{2}} = e^{\sqrt{2} \ln \pi}$$