Lesson 40 Fourier series converge!

Complex exponential: $e^{x+iy} = e^x e^{iy} = (e^x \cos y) + i(e^x \sin y) = e^z$

$z = x+iy$

Euler: $e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots$

$= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \cdots$

$= \underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right)}_{\cos y} + i\underbrace{\left(y - \frac{y^3}{3!} + \cdots\right)}_{\sin y}$

Fun thing: $E(x+iy) = (e^x \cos y) + i(e^x \sin y)$

1) $E'(z_0) = \underbrace{\lim_{z \to z_0} \frac{E(z) - E(z_0)}{z - z_0}}_{\text{complex derivative MA 425}} = E(z_0)$

2) $E(0) = 1$

$\Bigg\}$ determine $E(z)$

Play 8.3 games.

$E(z_1 + z_2) = E(z_1)E(z_2)$

$E(-z)E(z) = 1$. So $E(z)$ non-vanishing.

$E(-z) = 1/E(z)$

$E(x+iy) = E(x)E(iy)$

$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ R. of C. $= \infty$.

DeMoivre's formula $\quad (e^{i\theta})^N = e^{iN\theta} = \cos N\theta + i\sin N\theta$

$$\underbrace{e^{i\theta}e^{i\theta}\cdots e^{i\theta}}_{N\text{-times}} = e^{\overbrace{i\theta+i\theta+\cdots+i\theta}^{}_{N\text{-times}}} = e^{iN\theta} \quad \checkmark$$

Trig fcns:
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$(+)\quad e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$(-)\quad e^{i\theta}-e^{-i\theta} = 2i\sin\theta$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Fourier series: $\quad f(x) = \lim_{N\to\infty} S_N(x)$

where $\quad S_N(x) = a_0 + \sum_{n=1}^{N}\left(a_n\cos nx + b_n\sin nx\right)$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$S_N(x) = a_0 + \sum_{n=1}^{N} \left[a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right]$$

$$= a_0 + \sum_{n=1}^{N} \left[\left(\frac{a_n - ib_n}{2} \right) e^{inx} + \left(\frac{a_n + ib_n}{2} \right) e^{-inx} \right]$$

Write $c_n = \dfrac{a_n - ib_n}{2}$, $n = 1, 2 \ldots, N$

$c_{-n} = \dfrac{a_n + ib_n}{2}$, $n = 1, 2, \ldots, N$

$$\left[\begin{array}{l} S_N(x) = \displaystyle\sum_{-N}^{N} c_n e^{inx} \\ \text{where } \quad c_n = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \end{array} \right.$$

Check: $n > 0$: $\quad c_n = \dfrac{a_n - ib_n}{2} = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$- \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Use $\cos nx - i \sin nx = e^{-inx}$.

Similarly for $n < 0$. $\quad n = 0$ ✓

Wonderful thing $\quad S_N(x) = \displaystyle\sum_{n=-N}^{N} c_n e^{inx}$

$$= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt \right) e^{inx}$$

$$= \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-t)} \right)}_{D_N(x-t)} f(t) \, dt$$

$D_N(x-t)$ Dirichlet Kernel

Mind blowing thing: $D_N(t) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin t/2}$

Why: $D_N(t) = \frac{1}{2\pi} \left[\underbrace{e^{-iNt} + \cdots + e^{-it}}_{e^{-it}\left(1+e^{-it}+\cdots+e^{-i(N-1)t}\right)} + 1 + \underbrace{e^{it} + e^{i2t} + \cdots + e^{iNt}}_{e^{it}\left(1+e^{it}+\cdots+e^{i(N-1)t}\right)} \right]$

$1+r+\cdots+r^{N-1} = \frac{1-r^{(N-1)+1}}{1-r}$

$$e^{-it}\underbrace{\left(1+e^{-it}+\cdots+e^{-i(N-1)t}\right)}_{\frac{1-e^{-iNt}}{1-e^{-it}}} \qquad e^{it}\underbrace{\left(1+e^{it}+\cdots+e^{i(N-1)t}\right)}_{\frac{1-e^{iNt}}{1-e^{it}}}$$

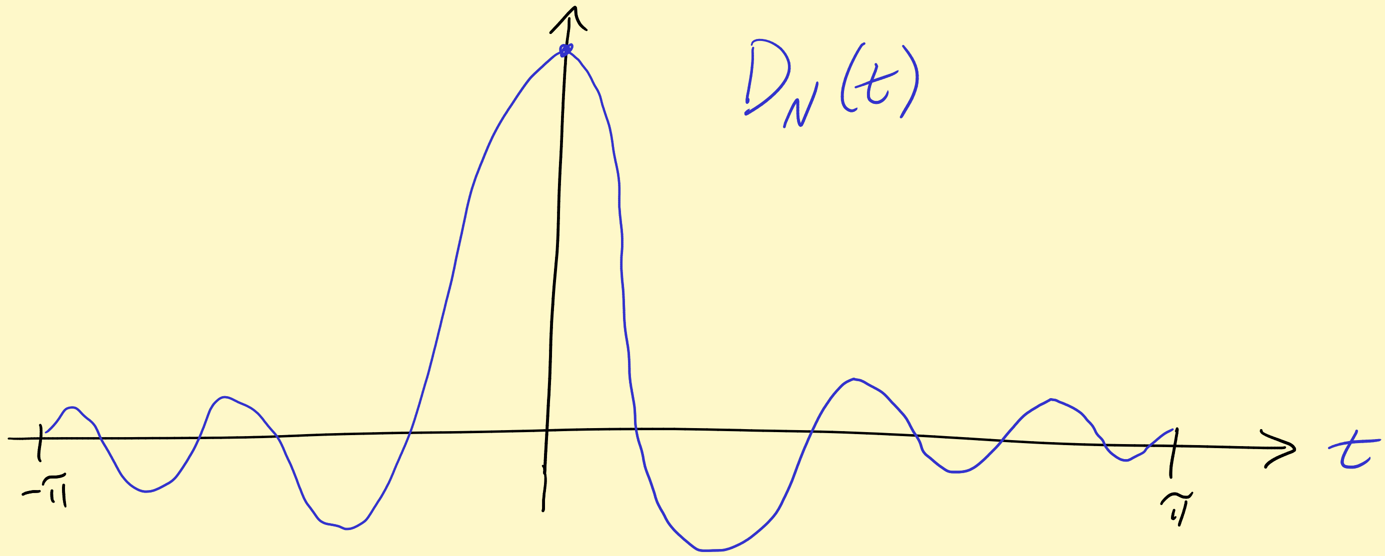
geom series!

$e^{int} = (e^{it})^n$

... algebra ... $= \frac{1}{2\pi} \cdot \frac{\left(\frac{e^{i(N+\frac{1}{2})t} - e^{-i(N+\frac{1}{2})t}}{2i} \right)}{\left(\frac{e^{it/2} - e^{-it/2}}{2i} \right)}$

$= \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin t/2}$ ✓

L'Hôpital's: $\lim_{t \to 0} D_N(t) = \frac{1}{2\pi} \frac{N+\frac{1}{2}}{1/2} = \frac{2N+1}{2\pi}$ Big!



Easy $\displaystyle\int_{-\pi}^{\pi} D_N(t)\,dt = 1$ because

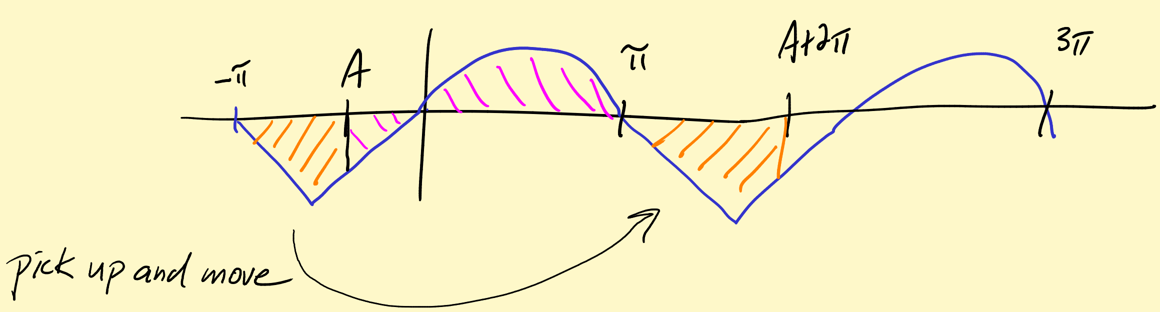
$$\int_{-\pi}^{\pi} e^{int}\,dt = 0 \quad \text{for } n \neq 0.$$

Fact: If f is C^2-smooth and 2π periodic, then $S_N(x) \longrightarrow f(x)$ uniformly on $[-\pi, \pi]$.

Lemma: If f is 2π-periodic, then

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{A}^{A+2\pi} f(t)\,dt \quad \text{for any } A.$$

Pf:



Pf of convergence:

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(\underbrace{x-t}_{\tau}) f(t)\,dt - f(x)$$

$$= \int_{x-\pi}^{x+\pi} D_N(\tau) f(x-\tau)\,d\tau - f(x)$$

Lemma: $$= \int_{-\pi}^{\pi} D_N(t) f(x-t)\,dt - \underbrace{f(x)}_{}$$

Aha! $1 = \int_{-\pi}^{\pi} D_N(t)\,dt$, so $\int_{-\pi}^{\pi} D_N(t) f(x)\,dt$

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t)\left[f(x-t) - f(x)\right] dt$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin t/2}\left[f(x-t) - f(x)\right] dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(N+\tfrac{1}{2})t \underbrace{\left[\frac{f(x-t) - f(x)}{\sin t/2}\right]}_{} dt$$

$f \in C'$: L'H shows $\lim_{t\to 0}$ exists.

f $\mathcal{C}^2$-smooth : Thing inside $[\quad]$ is $\mathcal{C}^1$-smooth!

Finally, integrate by parts. $\underbrace{\sin(N+\tfrac{1}{2})t \, dt}_{dv}$

$u = [\quad]$

A $\frac{1}{N+\frac{1}{2}}$ pops out! See whole thing $\to 0$ uniformly in x.

Advertisement for MA 425 : e^z

MA 428 : Fourier series

Wed, Fri Review.