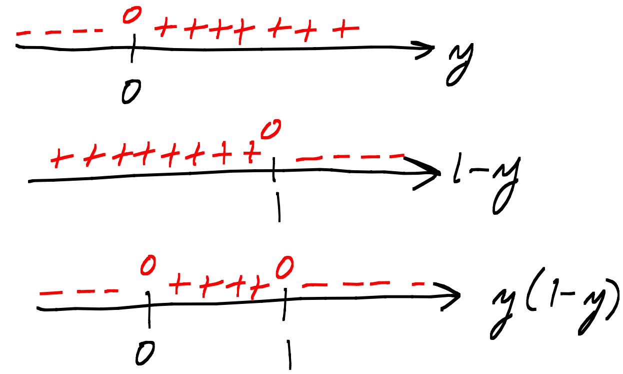
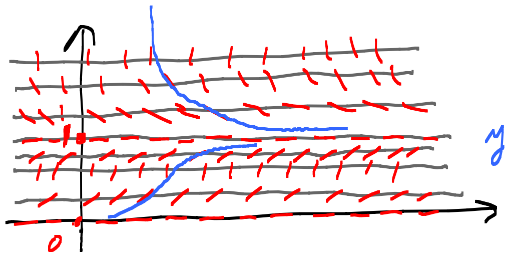
# Lesson 2 Separable equations

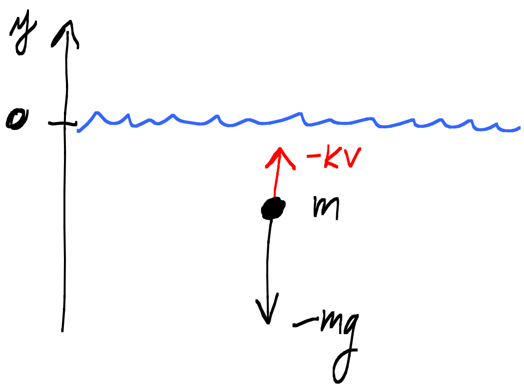
$$\frac{dy}{dt} = y(1-y)$$




$y \equiv 1$ is a sol$^n$

## Prob: Dropping a cannonball in ocean




Stokes' Law
Friction force $= -kv$

$$F = ma = m\frac{dv}{dt}$$

net force

$$\boxed{-mg - kv = m\frac{dv}{dt}}$$

$1^{st}$ order Linear ODE with constant coeff

Also separable.

Method:

$$\int \frac{m\,dv}{+mg + kv} = \int -dt$$

$$\int \underbrace{\frac{1}{g + \frac{k}{m}v}} dv = -t + C$$

$u = g + \frac{k}{m}v$

$du = \frac{k}{m}\,dv$

$dv = \frac{m}{k}\,du$

$$\int \frac{1}{u}\left(\frac{m}{k}\,du\right) = -t + C$$

$$\frac{m}{k}\ln\Big|\underset{\underset{g+\frac{k}{m}v}{\uparrow}}{u}\Big| = -t + C$$

$$e^{\ln\left|g+\frac{k}{m}v\right|} = e^{-\frac{kt}{m} + \underbrace{\frac{kC}{m}}_{\tilde{C}}}$$

Exponentiate!

$$\left|g+\frac{k}{m}v\right| = e^{-\frac{kt}{m}}\cdot e^{\tilde{C}}$$

So $$g+\frac{k}{m}v = \underbrace{\pm e^{\tilde{C}}}_{\tilde{\tilde{C}}}\cdot e^{-\frac{k}{m}t}$$

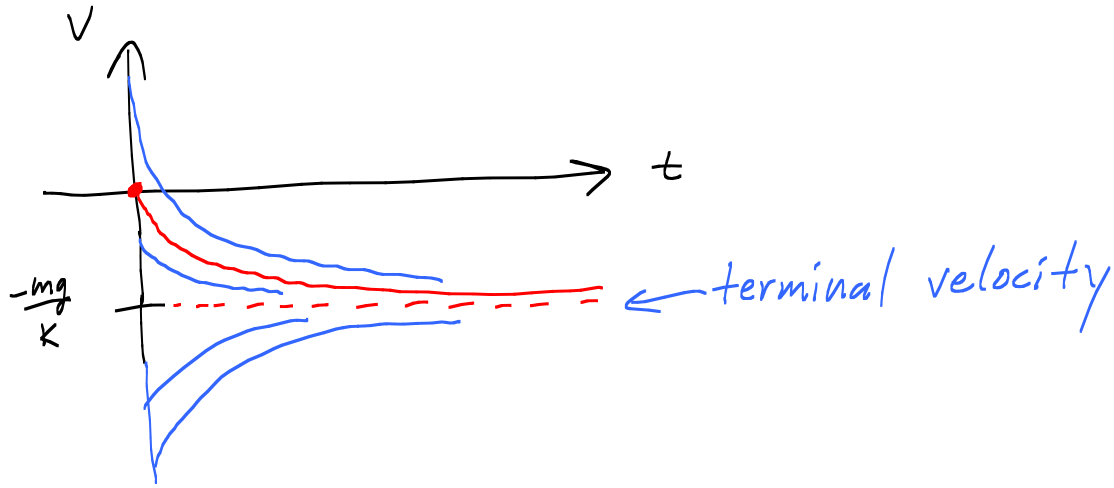
$$v = \frac{-mg}{k} + \underbrace{\frac{m\tilde{\tilde{C}}}{k}}_{C}\, e^{-\frac{k}{m}t}$$

$$\boxed{v = \underset{\underset{\text{steady state sol}^n}{\uparrow}}{\frac{-mg}{k}} + \underbrace{C e^{-\frac{k}{m}t}}_{\underset{\text{transient}}{\uparrow}}}$$

How to pin down C: Initial cond: $v(0) = 0 \leftarrow$ drop from rest

Box: Set $t=0$: $\underbrace{v(0)}_{0} = \frac{-mg}{k} + C\underbrace{e^{0}}_{1}$

So $\boxed{C = \frac{mg}{k}}$. $\quad V = \frac{-mg}{k} + \frac{mg}{k} e^{-\frac{k}{m}t}$




Separable eqns: Implicit diff'tion

$$x^2 + y^2 = C$$ defines fcn $y(x)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Method:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Let $F(x) = \int f(x)\,dx$

$G(y) = \int g(y)\,dy$

(Pick an antiderivative)

$$\int g(y)\,dy = \int f(x)\,dx$$

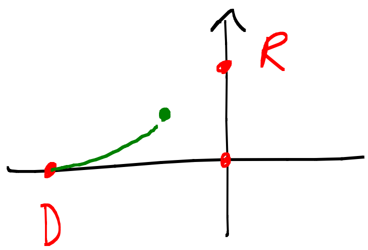
$$G(y) = F(x) + C$$ defines $y(x)$ implicitly and solves ODE!

Really? $\underbrace{\frac{d}{dx} G(y)}_{} = F'(x) + 0$

$$G'(y)\frac{dy}{dx} = F'(x)$$

$$g(y)\frac{dy}{dx} = f(x) \qquad \text{Yes!}$$

EX:




$$-x\frac{d^2y}{dx^2} = \frac{V_R}{V_D}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Let $p = \frac{dy}{dx}$

Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

ODE $-x\frac{dp}{dx} = \frac{V_R}{V_D}\sqrt{1+p^2} \longleftarrow$ Separable!

$$\underbrace{\int \frac{dp}{\sqrt{1+p^2}}}_{} = -\int \frac{V_R}{V_D}\frac{1}{x}\,dx$$

$$\sinh^{-1} p = \frac{-V_R}{V_D}\ln|x| + C$$

So $\underbrace{p}_{\frac{dy}{dx}} = \sinh\left(-\frac{V_R}{V_D}\ln|x| + C\right)$

Hmmm. $\sinh u = \frac{1}{2}\left(e^{u} - e^{-u}\right)$

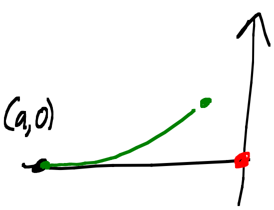
$$e^{-\frac{v_R}{v_D}\ln|x| + C}$$

$$\underbrace{e^{\ln|x|^{-v_R/v_D}}}_{|x|^{-v_R/v_D}} \quad e^{C}$$

Last step: $\frac{dy}{dx} = \text{fcn of } x$.

Get $y$: Integrate. Done.

Arbitrary constants?




$\frac{dy}{dx}(a) = 0$

$y(a) = 0$